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Modified Szász-Kantorovich type operators with two parameters using Charlier polynomials

Abdul Wafi, Nadeem Rao

Department of Mathematics, Jamia Millia Islamia, New Delhi-110 025, India
E-mail: awafi@jmi.ac.in, nadeemrao1990@gmail.com

Abstract: The aim of this article is to introduce the Kantorovich form of generalized Szász-type operators involving Charlier polynomials with certain parameters. In this paper, we discuss the rate of convergence, better error estimates and Korovkin-type theorem in polynomial weighted space. Further, we investigate the local approximation results with the help of Ditzian-Totik modulus of smoothness, second order modulus of continuity, Peetre's K-functional and Lipschitz class.

Keywords: Szász operators, Charlier polynomials, Ditzian-Totik modulus of smoothness, Peetre's K-functional, Lipschitz class.
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1. Introduction

Approximation theory plays an important role in mathematical analysis and other branches of mathematics. The results of theory of approximation is generally related to positive linear operators, and deals with rate of convergence and order of approximation. Weierstrass was the first who gave an important theorem, namely, Weierstrass approximation theorem in this regard. The aim of this theorem is to minimize the maximum value of $|f(x)-P_n(x)|$ for the continuous functions f on $[a,b]$, where $P_n(x)$ is a sequence of polynomials of degree n. The proof of this theorem was considered very difficult until Bernstein gave an elegant and simple proof of it. Bernstein [1] defined the positive linear operators using binomial distribution in the following way

$$B_n(f;x)=\sum_{k=0}^{n}P_{n,k}(x)\,f\left(\frac{k}{n}\right),\quad n=1,2,3...,k=0,1,2,... \tag{1}$$

where $P_{n,k}(x)={}^{n}c_k x^k(1-x)^{n-k}$, $x\in[0,1]$ and proved pointwise and uniform approximation in the space of continuous functions on $[0,1]$. These operators provide the powerful tool for numerical analysis, computer added geometric design (CAGD) and solutions of differential equations. But these operators are not suitable for discontinuous functions. Later on, Kantorovich [2] generalized the Bernstein operators for integrable functions as

$$K_n(f;x)=(n+1)\sum_{k=0}^{n}P_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(t)dt,\quad k=0,1,2,3,...,n=1,2,3,.... \tag{2}$$

Szász [3] introduced linear positive operators in the sense of exponential growth on non-negative semi axes

$$S_{n,k}(f;x)=\sum_{k=0}^{\infty} s_{n,k}(x)\, f(\frac{k}{n}), \quad n=1,2,3,.... \tag{3}$$

where $s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}$, $f\in C[0,\infty)$ and bounded on $(0,\infty)$. Several generalizations of these operators have been studied by different researchers ([4]-[6]). A generalization of operators (1) was given by Stancu [7] depending on the parameters α and β such that $0\le\alpha\le\beta$ on $[0,1]$. Many operators preserve the constant and linear functions but these operators do not preserve x^2. King [8] introduced a method in order to preserve x^2 for the Bernstein operators.

Recently, Varma and Tasdelen [9] gave a generalization of well known Szász-Mirakjan operators using Charlier polynomials [10] having the generating function of the form

$$e^t(1-\frac{t}{a})^u=\sum_{k=0}^{\infty} C_k^{(a)}(u)\frac{t^k}{k!}, \quad |t|<a \tag{4}$$

and the explicit representation

$$C_k^{(a)}(u)=\sum_{r=0}^{k} \genfrac{}{}{0pt}{}{k}{r} (-u)_r(\frac{1}{a})^r,$$

where $(\alpha)_k$ is the Pochhammer's symbol given by

$$(\alpha)_0=1,(\alpha)_k=\alpha(\alpha+1)...(\alpha+k-1),k=1,2,....$$

We note that for $a>0$ and $u\le 0$, Charlier polynomials are positive. Varma and Tadelen [9] defined the Szász-type and Kantorovich-Szász-type operators operators as

$$L_n(f;x,a)=e^{-1}(1-\frac{1}{a})^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nx)}{k!}\, f(\frac{k}{n}), \tag{5}$$

$$L_n^*(f;x,a)=ne^{-1}(1-\frac{1}{a})^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \tag{6}$$

where $a>1$, $n=1,2,3...,k=0,1,2...$ and $x\ge 0$.

Motivated by the above development, we define a Kantorovich version of $T_{n,a}$ with two parameters $0\le\alpha\le\beta$ as

$$K_{n,a}^{\alpha,\beta}(f;r_{n,a}^*(x;\alpha,\beta),a)= \tag{7}$$

$$(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^*(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}^*(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{(n+\beta)}}^{\frac{k+\alpha+1}{(n+\beta)}} f(s)ds,$$

where

$$r_{n,a}^{*}(x;\alpha,\beta)=\frac{-(4+2\alpha+\frac{1}{a-1})+\sqrt{(4+2\alpha+\frac{1}{a-1})^{2}+4((n+\beta)^{2}x^{2}-\frac{10}{3}-3\alpha-\alpha^{2})}}{2n}. \qquad (8)$$

We observe that

(i) if $\alpha=\beta=0$ and $r_{n,a}^{*}(x;\alpha,\beta)=x$, operators (7) reduce to operators (6),
and

(ii) for $\alpha=\beta=0$ and $r_{n,a}^{*}(x;\alpha,\beta)=x$ as $a\to\infty$ and taking $x-\frac{1}{n}$ instead of x, operators (7) reduce to the Classical Kantorovich-Szász operators.

In the present paper, we discuss the rate of convergence for continuous functions, first order derivative of the function and weighted Korovkin type theorem. Further, we investigate some direct and local approximation results using Ditzian-Totik modulus of smoothness, second order modulus of continuity, Peetre's K-functional and Lipschitz space.

2. Basic Estimates

Lemma 2.1. Let $e_i=t^i, i=0,1,2$. Then for the operators $K_{n,a}^{\alpha,\beta}$, we have

$$(i)\quad K_{n,a}^{\alpha,\beta}(1;x)=1,$$

$$(ii)\quad K_{n,a}^{\alpha,\beta}(t;x)=\frac{-(1+\frac{1}{a-1})+\sqrt{-(4+2\alpha+\frac{1}{a-1})^{2}+4((n+\beta)^{2}x^{2}-\frac{10}{3}-3\alpha-\alpha^{2})}}{2(n+\beta)},$$

$$(iii)\quad K_{n,a}^{\alpha,\beta}(t^{2};x)=x^{2}.$$

Proof. $(i)\quad K_{n,a}^{\alpha,\beta}(1;x,a)=(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}1.dt$

$$=(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))}{k!}[\frac{k+\alpha+1}{n+\beta}-\frac{k+\alpha}{n+\beta}]$$

$$=(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}e(1-\frac{1}{a})^{-(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\times\frac{1}{n+\beta}$$

$$=1,$$

(ii) $K_{n,a}^{\alpha,\beta}(t;x)=(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))}{k!}$

$$x\frac{1}{2}[(\frac{k+\alpha+1}{n+\beta})^2-(\frac{k+\alpha}{n+\beta})^2]$$

$$=(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr^*_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^*_{n,a}(x;\alpha,\beta))}{k!}\frac{2(k+\alpha)+1}{2(n+\beta)^2}$$

$$=e^{-1}(1-\frac{1}{a})^{(a-1)nr^*_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^*_{n,a}(x;\alpha,\beta))}{k!}\frac{(k+\alpha)}{(n+\beta)}+\frac{1}{2(n+\beta)}.$$

$$=\frac{1}{n+\beta}e^{-1}(1-\frac{1}{a})^{(a-1)nr^*_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^*_{n,a}(x;\alpha,\beta))}{k!}k+\frac{2\alpha+1}{2(n+\beta)}$$

$$=\frac{1}{n+\beta}e^{-1}(1-\frac{1}{a})^{(a-1)nr^*_{n,a}(x;\alpha,\beta)}e(1-\frac{1}{a})^{-(a-1)nr^*_{n,a}(x;\alpha,\beta)}(1+nr^*_{n,a}(x;\alpha,\beta))+\frac{2\alpha+1}{2(n+\beta)}$$

$$=\frac{nr^*_{n,a}(x;\alpha,\beta)}{2(n+\beta)}+\frac{2\alpha+3}{2(n+\beta)}$$

$$=\frac{-(4+2\alpha+\frac{1}{a-1})+\sqrt{(4+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-\frac{10}{3}-3\alpha-\alpha^2)}}{2(n+\beta)}+\frac{2\alpha+3}{2(n+\beta)}$$

$$=\frac{-(1+\frac{1}{a-1})+\sqrt{(4+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-\frac{10}{3}-3\alpha-\alpha^2)}}{2(n+\beta)}.$$

Similarly, we can prove

$$K^{\alpha,\beta}_{n,a}(t^2;x,a)=x^2.$$

Lemma 2.2. Let $\psi_x^i(t)=(t-x)^i, i=0,1,2$. Then

$$K^{\alpha,\beta}_{n,a}(\psi_x^0;x)=1,$$

$$K^{\alpha,\beta}_{n,a}(\psi_x^1;x)=-\frac{(1+\frac{1}{a-1})}{2(n+\beta)}+\frac{\frac{8}{3}+4\alpha+\frac{4\alpha+8}{a-1}}{(n+\beta)(\sqrt{(4+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-\frac{10}{3}-3\alpha-\alpha^2)}+2(n+\beta)x)},$$

$$K^{\alpha,\beta}_{n,a}(\psi_x^2;x)=\frac{(1+\frac{1}{a-1})}{n+\beta}x-\frac{x}{(n+\beta)}\frac{(\frac{8}{3}+4\alpha+\frac{4\alpha+8}{a-1})}{\sqrt{(4+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-\frac{10}{3}-3\alpha-\alpha^2)}+2(n+\beta)x}.$$

Proof. In view of Lemma 2.1 and linearity property we can easily prove this Lemma.

3. Order of approximation for the function f and derivative of f

Let $f \in C[0,\infty)$. Then modulus of continuity of uniformly countinuous functions on $[0,\infty)$ is defined by

$$\omega(f;\delta) = \sup_{|t-y|\le\delta} |f(t)-f(y)|, \quad t, y \in [0,\infty).$$

For $f \in C[0,\infty)$ and $\delta > 0$, one has

$$|f(t)-f(y)| \le (1+\frac{(t-y)^2}{\delta^2})\omega(f;\delta). \tag{9}$$

And

$$E = \{f:[0,\infty) \to R, |f(x)| \le Me^{Ax}, A \in R \text{ and } M \in R^+\}.$$

Theorem 3.1. *Let* $f \in C[0,\infty) \cap E$ *and* $x \ge 0$. *Then for operators* $K_{n,a}^{\alpha,\beta}$, *we have*

$$|K_{n,a}^{\alpha,\beta}(f;x) - f(x)| \le 2\omega(f;\delta_{n,a}^{\alpha,\beta}),$$

where $\delta_{n,a}^{\alpha,\beta} = \sqrt{K_{n,a}^{\alpha,\beta}(\psi_x^2;x)}$.

Proof. From (9), we have

$$|K_{n,a}^{\alpha,\beta}(f;x) - f(x)| \le (n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^*(x;\alpha,\beta)} \sum_{k=0}^{\infty} \frac{C_k^{(u)}(-(a-1)nr_{n,a}^*(x;\alpha,\beta))}{k!} \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |f(t)-f(x)|\,dt$$

$$\le \{(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^*(x;\alpha,\beta)} \sum_{k=0}^{\infty} \frac{C_k^{(u)}(-(a-1)nr_{n,a}^*(x;\alpha,\beta))}{k!} \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} (1+\frac{(t-x)^2}{(\delta_{n,a}^{\alpha,\beta})^2})dt\}$$

$$\times\omega(f;\delta_{n,a}^{\alpha,\beta})$$

$$\le \{1+\frac{K_{n,a}^{\alpha,\beta}(\psi_x^2;x)}{(\delta_{n,a}^{\alpha,\beta})^2}\}\omega(f;\delta_{n,a}^{\alpha,\beta})$$

$$= 2\omega(f;\delta_{n,a}^{\alpha,\beta}),$$

where $\delta_{n,a}^{\alpha,\beta} = \sqrt{K_{n,a}^{\alpha,\beta}(\psi_x^2;x)}$.

Remark. For the Kantorovich-Szász type operators L_n^* given by (6), we have, for every $f \in C[0,\infty) \cap E$,

$$|L_n^*(f;x,a) - f(x)| \le 2\omega(f;\delta), \tag{10}$$

where $\delta = \sqrt{\frac{x}{n}(1+\frac{1}{a-1})+\frac{10}{3n^2}}$. Here we show that our operators $K_{n,a}^{\alpha,\beta}$ has the better approximation than the operators L_n^*.

Since

$$\frac{x}{n+\beta}(1+\frac{1}{a-1})<\frac{x}{n}(1+\frac{1}{a-1})+\frac{10}{3n^2}.$$

Then $\delta_{n,a}^{\alpha,\beta}<\delta$.

Theorem 3.2. *If* $f\in C[0,\infty)$ *and bounded on* $[0,\infty)$ *has continuous derivative and* $\omega_1(f;\delta_{n,\beta})$ *is the modulus of continuity of* $f'(x)$ *, then, for* $0\le\alpha\le\beta, a>1$ *and* $x\in[0,b], b<\infty$ *, we have*

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)|\le\omega_1(f;(n+\beta)^{-1})\sqrt{K_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}\{1+\sqrt{(n+\beta)}\sqrt{K_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}\}.$$

Proof. It is known that

$$\begin{aligned} f(x_1)-f(x_2)&=(x_1-x_2)f'(\xi)\\ &=(x_1-x_2)f'(x_1)+(x_1-x_2)[f'(\xi)-f'(x_1)], \end{aligned} \tag{11}$$

for $x_1,x_2\in[0,b]$ and $x_1<\xi<x_2$. Also, we have

$$|(x_1-x_2)[f'(\xi)-f'(x_1)]|\le|x_1-x_2|(\lambda+1)\omega_1(\delta),\quad \lambda=\lambda(x_1,x_2;\delta). \tag{12}$$

Next, we find

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)|=. \tag{13}$$

$$|(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}f(t)-f(x)dt|$$

Using (11) and (12), we get

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)|\le|(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}(x-t)f'(x)dt|$$

$$+\omega_1(\delta_n^{\beta})(\lambda+1)(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|t-x|dt$$

$$\le\omega_1(\delta_n^{\beta})\{e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|t-x|dt$$

$$+e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{\lambda\ge1}C_k^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|t-x|\lambda(x,t;\delta)dt\}$$

$$\le \omega_1(\delta_n^\beta)\{e^{-1}(1-\frac{1}{a})^{(a-1)nr^*_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^*_{n,a}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|t-x|\,dt$$

$$+\frac{1}{\delta_n^\beta}e^{-1}(1-\frac{1}{a})^{(a-1)nr^*_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^*_{n,a}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}(t-x)^2\,dt\}$$

$$\le \omega_1(\delta_n^\beta)(\sqrt{K_{n,a}^{\alpha,\beta}(\psi_x^2;x)}+\frac{K_{n,a}^{\alpha,\beta}(\psi_x^2;x)}{\delta_n^\beta})$$

$$=\omega_1(\delta_n^\beta)\sqrt{K_{n,a}^{\alpha,\beta}(\psi_x^2;x)}\{1+\frac{\sqrt{K_{n,a}^{\alpha,\beta}(\psi_x^2;x)}}{\delta_n^\beta}\}$$

Taking $\delta_n^\beta=(n+\beta)^{-1}$, we get

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)|\le \omega_1(f;(n+\beta)^{-1})\sqrt{K_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}\{1+\sqrt{(n+\beta)}\sqrt{K_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}\}.$$

We shall now discuss the Korovkin-type theorem in polynomial weighted space of continuous and unbounded functions defined on $[0,\infty)$. Here we recall some symbols and notions from [11]. Let $\rho(x)=1+x^2$, $-\infty<x<\infty$ and $B_\rho[0,\infty)=\{f(x):|f(x)|\le M_f\rho(x),\rho(x)$ is weight function, M_f is a constant depending on f and $x\in[0,\infty)\}$. $C_\rho[0,\infty)$ is the space of continuous function in $B_\rho[0,\infty)$ with the norm $\|f\|_\rho=\sup\limits_{x\in[0,\infty)}\frac{|f(x)|}{\rho(x)}$ and

$C_\rho^k=\{f\in C_\rho:\lim\limits_{|x|\to\infty}\frac{f(x)}{\rho(x)}=k,$ where k is a constant depending on $f\}$.

Theorem 3.3. *Let* $f\in C_\rho^k$. *Then, the operators* $K_{n,a}^{\alpha,\beta}(f;x)$ *converges uniformly to* $f(x)$ *on* $[0,\infty)$.

Proof. By the Korovkin Theorem, it is sufficient to verify that

$$\lim_{n\to\infty}\|K_{n,a}^{\alpha,\beta}(t^i;x)-x^i\|_\rho=0,\quad for\quad i=0,1,2.$$

It is obvious that $\lim\limits_{n\to\infty}\|K_{n,a}^{\alpha,\beta}(1;x)-1\|_\rho=0$ and $\lim\limits_{n\to\infty}\|K_{n,a}^{\alpha,\beta}(x^2;x)-x^2\|_\rho=0$. Also, we have $\sup\limits_{x\in[0,\infty)}\frac{|K_{n,a}^{\alpha,\beta}(t;x)-x|}{1+x^2}=\sup\limits_{x\in[0,\infty)}\frac{\left|\frac{-(1+\frac{1}{a-1})+\sqrt{(4+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-\frac{10}{3}-3\alpha-\alpha^2)}}{2(n+\beta)}-x\right|}{1+x^2}$

$$= \sup_{x\in[0,\infty)} \frac{\left| -\frac{(1+\frac{1}{a-1})}{2(n+\beta)} + \frac{\frac{8}{3}+4\alpha+\frac{4\alpha+8}{a-1}}{2(n+\beta)\sqrt{(4+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-\frac{10}{3}-3\alpha-\alpha^2)}+2(n+\beta)x} \right|}{1+x^2}$$

$$\leq \sup_{x\in[0,\infty)} \left| \frac{\frac{4}{3}+2\alpha+\frac{2\alpha+4}{a-1}}{(n+\beta)\sqrt{(4+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-\frac{10}{3}-3\alpha-\alpha^2)}+2(n+\beta)x} \frac{1}{(1+x^2)} \right|$$

$$+ \sup_{x\in[0,\infty)} \left| \frac{(1+\frac{1}{a-1})}{2(n+\beta)} \frac{1}{(1+x^2)} \right|$$

which shows that $\| K_{n,a}^{\alpha,\beta}(t;x)-x \|_{\rho} \to 0$. as $n\to\infty$,

Hence, the theorem is proved.

4. Direct Estimate

Ditzian-Totik Modulus of smoothness [12] is defined as:

$$\omega_{\varphi^\lambda}^2(f;\delta) = \sup_{0<h\leq\delta} \| \Delta_{h\varphi(x)}^2 f(x) \|,$$

$$= \sup_{0<h\leq\delta} \sup_{x\pm h\varphi^\lambda\in[0,\infty)} | f(x-h\varphi^\lambda(x)) - 2f(x) + f(x+h\varphi^\lambda(x)) |,$$

where $\varphi^2(x)=x$. And, Peetre's K-functional [12] is given by

$$K_{\varphi^\lambda}(f,\delta^2) = \inf_g (\| f-g \|_{C[0,\infty)} + \delta^2 \| \varphi^2 \lambda g'' \|_{C[0,\infty)}), \quad g, g' \in AC_{loc}. \tag{14}$$

The K-functional is equivalent to the modulus of smoothness, i.e.,

$$C^{-1}K_{\varphi^\lambda}(f,\delta^2) \leq \omega_{\varphi^\lambda}^2(f,\delta) \leq CK_{\varphi^\lambda}(f,\delta^2). \tag{15}$$

First result based on Ditziaz-Totik modulus of smoothness was given by Ditzian [13] for the Bernstein polynomials as:

$$| B_n(f;x) - f(x) | \leq C\omega_{\varphi^\lambda}^2(f, n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}).$$

Now, we prove the similar result for the operator $K_{n,a}^{\alpha,\beta}$.

Theorem 4.1. *For* $f\in L_p(0,\infty), 0\leq p<\infty$, *and* $a>1$, *we have*

$$| K_{n,a}^{\alpha,\beta}(f;x) - f(x) | \leq C\omega_{\varphi^\lambda}^2(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}) \quad for \quad large \quad n$$

where $0\leq\lambda\leq 1$, $\varphi^2(x)=x$.

Proof. Using (12),(13), we have

$$\| f-g \|_{L_p(0,\infty)} \leq A\omega_{\varphi^\lambda}^2(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}), \tag{16}$$

$$(n+\beta)^{-1}\varphi(x)^{2-2\lambda}\,\|\varphi^{2\lambda}g''\|_{L_p(0,\infty)}\leq B\omega^2_{\varphi^\lambda}(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}). \tag{17}$$

Next, we can choose $g_n\equiv g_{n,x,\lambda}$ for fixed x and $\lambda+1$ such that

$$|K^{\alpha,\beta}_{n,a}(f;x)-f(x)|\leq|K^{\alpha,\beta}_{n,a}(f-g_n;x)-(f-g_n)(x)|+|K^{\alpha,\beta}_{n,a}(g_n;x)-g_n(x)|,$$
$$\leq 2\|f-g_n\|_{L_p(0,\infty)}+|K^{\alpha,\beta}_{n,a}(g_n;x)-g_n(x)|.$$

From (14), we get

$$|K^{\alpha,\beta}_{n,a}(f;x)-f(x)|\leq 2A\omega^2_{\varphi^\lambda}(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda})+|K^{\alpha,\beta}_{n,a}(g_n;x)-g_n(x)|. \tag{18}$$

Now, the last term can be calculated by using Taylor's formula

$$|K^{\alpha,\beta}_{n,a}(g_n(t)-g_n(x);x)|\leq|g'_n(x)K^{\alpha,\beta}_{n,a}((t-x);x)|+|K^{\alpha,\beta}_{n,a}(\int_t^x(x-u)g''_n(u)du;x)|$$

$$\leq K^{\alpha,\beta}_{n,a}\Big(\frac{|x-\frac{k}{n}|}{\varphi^{2\lambda}(x)}\int_{\frac{k}{n}}^{x}\varphi^{2\lambda}(u)\,|g''_n(u)du|;x\Big)$$

$$\leq\|\varphi^{2\lambda}g''_n\|_{L_p(0,\infty)}\frac{1}{\varphi^{2\lambda}(x)}K^{\alpha,\beta}_{n,a}((t-x)^2;x)$$

$$\leq\|\varphi^{2\lambda}g''_n\|_{L_p(0,\infty)}\frac{1}{\varphi^{2\lambda}(x)}\frac{x}{n+\beta}\frac{(n+\beta)K^{\alpha,\beta}_{n,a}((t-x)^2;x)}{x}$$

$$\leq\|\varphi^{2\lambda}g''_n\|_{L_p(0,\infty)}\frac{x(n+\beta)^{-1}}{\varphi^{2\lambda}(x)}\frac{(n+\beta)K^{\alpha,\beta}_{n,a}((t-x)^2;x)}{x}.$$

For sufficiently large value of n, we get

$$\frac{(n+\beta)K^{\alpha,\beta}_{n,a}((t-x)^2;x)}{x}\leq(1+\frac{1}{a-1}).$$

Therefore

$$|K^{\alpha,\beta}_{n,a}(g_n(t)-g_n(x);x)|\leq(1+\frac{1}{a-1})B\omega^2_{\varphi^\lambda}(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}. \tag{19}$$

Using (16) and (17), we get

$$|K^{\alpha,\beta}_{n,a}(f(t)-f(x);x)|\leq M\omega^2_\lambda(f,(n+\beta)^{\frac{-1}{2}}\varphi(x)^{1-\lambda})$$

where $M=max(2A,(1+\frac{1}{a-1})B)$.

Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions f on $[0,\infty)$ endowed with the norm

$$\| f \| = \sup_{0\le x<\infty} | f(x) |.$$

Then, for any $\delta > 0$, Peeter's K-functional is defined as

$$K_2(f,\delta) = inf\{\| f-g \| + \delta \| g'' \| : g \in C_B^2[0,\infty)\},$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By Devore and Lorentz [[14], p.177, Theorem 2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}),$$

where $\omega_2(f;\delta)$ is the second order modulus of continuity is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{x\in[0,\infty)} | f(x+2h) - 2f(x+h) + f(x) |.$$

Theorem 4.2. *Let* $f \in C_B^2[0,\infty)$. *Then for all* $x \in [0,\infty)$ *there exists a constant* $K > 0$ *such that*

$$| K_{n,a}^{\alpha,\beta}(f;x) - f(x) | \le K\omega_2(f;\sqrt{\Pi_{n,a}^{\alpha,\beta}(x)}) + \omega(f;\Lambda_{n,a}^{\alpha,\beta})$$

where $\Lambda_{n,a}^{\alpha,\beta}(x) = K_{n,a}^{\alpha,\beta}(\psi_x;x)$ and $\Pi_{n,a}^{\alpha,\beta}(x) = K_{n,a}^{\alpha,\beta}(\psi_x^2;x) + (K_{n,a}^{\alpha,\beta}(\psi_x;x))^2$.

Proof. First, we define the auxiliary operators

$$K_{n,a}^{\alpha,\beta}(f;x) = K_{n,a}^{\alpha,\beta}(f;x) + f(x) - f(\Lambda_{n,a}^{\alpha,\beta}(x)) \qquad (20)$$

We find that

$$K_{n,a}^{\alpha,\beta}(1;x) = 1,$$

$$K_{n,a}^{\alpha,\beta}(\psi_x(t);x) = 0$$

$$| K_{n,a}^{\alpha,\beta}(f;x) | \le 3\| f \|. \qquad (21)$$

Let $g \in C_B^2[0,\infty)$. By the Taylor's theorem

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv, \qquad (22)$$

Now

$$\hat{K}_{n,a}^{\alpha,\beta}(g;x) - g(x) = g'(x)\hat{K}_{n,a}^{\alpha,\beta}(t-x;x) + \hat{K}_{n,a}^{\alpha,\beta}(\int_x^t (t-v)g''(v)dv;x)$$

$$= \hat{K}_{n,a}^{\alpha,\beta}(\int_x^t (t-v)g''(v)dv;x)$$

$$= K_{n,a}^{\alpha,\beta}(\int_x^t (t-v)g''(v)dv;x) - \int_x^{\Lambda_{n,a}^{\alpha,\beta}} (\Lambda_{n,a}^{\alpha,\beta} - v)g''(v)dv.$$

Therefore

$$|\hat{K}_{n}^{\alpha,\beta}(g;x)-g(x)|\leq|K_{n,a}^{\alpha,\beta}(\int_{x}^{t}(t-v)g''(v)dv;x)|+|\int_{x}^{\Lambda_{n,a}^{\alpha,\beta}}(\Lambda_{n,a}^{\alpha,\beta}-v)g''(v)dv|.$$

Since

$$|\int_{x}^{t}(t-v)g''(v)dv|\leq(t-x)^{2}\|g''\| \tag{23}$$

and

$$|\int_{x}^{\Lambda_{n,a}^{\alpha,\beta}}(\Lambda_{n,a}^{\alpha,\beta}-v)g''(v)dv|\leq(\Lambda_{n,a}^{\alpha,\beta})^{2}\|g''\| \tag{24}$$

Then from (22), (23)and (24) implies that

$$\begin{aligned}|\hat{K}_{n,a}^{\alpha,\beta}(g;x)-g(x)|&\leq\{K_{n,a}^{\alpha,\beta}((t-x)^{2};x)+(\Lambda_{n,a}^{\alpha,\beta})^{2}\}\|g''\|\\&=\Pi_{n,a}^{\alpha,\beta}(x)\|g''\| \end{aligned}\tag{25}$$

Next, we have

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)|\leq|\hat{K}_{n,a}^{\alpha,\beta}(f-g;x)|+|(f-g)(x)|+|\hat{K}_{n,a}^{\alpha,\beta}(g;x)-g(x)|+|f(\Lambda_{n,a}^{\alpha,\beta})-f(x)|$$

Using (25), we have

$$\begin{aligned}|K_{n,a}^{\alpha,\beta}(f;x)-f(x)|&\leq 4\|f-g\|+\hat{K}_{n,a}^{\alpha,\beta}(g;x)-g(x)|+|f(\Lambda_{n,a}^{\alpha,\beta})-f(x)|\\&\leq 4\|f-g\|+\Pi_{n,a}^{\alpha,\beta}(x)\|g''\|+\omega(f;\Lambda_{n,a}^{\alpha,\beta}).\end{aligned}$$

By the definition of Peetre’s K-functional

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)|\leq C\omega_{2}(f;\sqrt{\gamma_{n}^{\alpha,\beta}(x)})+\omega(f;\Lambda_{n,a}^{\alpha,\beta}).$$

Here, we discuss a local result in Lipschitz class

$$Lip_{M}^{*}=\{f\in C[0,\infty):|f(t)-f(x)|\leq M\frac{|t-x|^{\alpha}}{(t+x)^{\frac{\alpha}{2}}}:x,t\in(0,\infty)\}$$

where M is a constant and $0<\alpha\leq 1$ to prove the following theorem:

Theorem 4.3 *Let* $f\in Lip_{M}^{*}(\alpha)$ *and* $x\in(0,\infty)$. *Then, we have*

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)|\leq M[\frac{\Theta_{n}^{\alpha,\beta}(x)}{x}],$$

where $\Theta_{n,a}^{\alpha,\beta}(x)=K_{n,a}^{\alpha,\beta}((t-x)^{2};x)$.

Proof. Let $\alpha=1$ and $x\in(0,\infty)$. Then, for $f\in Lip_{M}^{*}(1)$, we have

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)|\leq(n+\beta)e^{-1}$$

$$(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_{k}^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|f(t)-f(x)|dt$$

$$\leq M(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr^{*}_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^{*}_{n,a}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}\frac{|t-x|}{\sqrt{t+x}}dt.$$

$$\leq \frac{M}{\sqrt{x}}(n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr^{*}_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^{*}_{n,a}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|t-x|\,dt$$

$$\leq \frac{M}{\sqrt{x}}K_n^{\alpha,\beta}(|t-x|;x)$$

$$\leq M\frac{\sqrt{K_n^{\alpha,\beta}((t-x)^2;x)}}{\sqrt{x}}$$

$$= M(\frac{\Theta_{n,a}^{\alpha,\beta}(x)}{x})^{\frac{1}{2}}$$

Thus, the assertion hold for $\alpha = 1$. Now, we will prove for $\alpha \in (0,1)$. From the Holder inequality with $p = \frac{1}{\alpha}, q = \frac{1}{1-\alpha}$, we have

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)| = (e^{-1}(1-\frac{1}{a})^{(a-1)nr^{*}_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^{*}_{n,a}(x;\alpha,\beta))}{k!}((n$$

$$+\beta)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|f(t)-f(x)|\,dt)^{\frac{1}{\alpha}})^{\alpha}$$

$$\times(e^{-1}(1-\frac{1}{a})^{(a-1)nr^{*}_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^{*}_{n,a}(x;\alpha,\beta))}{k!})^{1-\alpha}$$

$$\leq (e^{-1}(1-\frac{1}{a})^{(a-1)nr^{*}_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^{*}_{n,a}(x;\alpha,\beta))}{k!}((n+\beta)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|f(t)-f(x)|\,dt)^{\frac{1}{\alpha}})^{\alpha}$$

Since $f \in Lip_M^{*}\alpha$, we obtain

$$|K_{n,a}^{\alpha,\beta}(f;x)-f(x)| \leq M((n+\beta)e^{-1}(1$$

$$-\frac{1}{a})^{(a-1)nr^{*}_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr^{*}_{n,a}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}\frac{|t-x|}{\sqrt{t+x}}dt)^{\alpha}$$

$$\leq \frac{M}{x^{\frac{\alpha}{2}}}((n+\beta)e^{-1}(1-\frac{1}{a})^{(a-1)nr_{n,a}^{*}(x;\alpha,\beta)}\sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}^{*}(x;\alpha,\beta))}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|t-x|\,dt)^{\alpha}$$

$$= \frac{M}{x^{\frac{\alpha}{2}}}(K_{n,a}^{\alpha,\beta}(|t-x|;x))^{\alpha}$$

$$\leq M(\frac{\Theta_{n,a}^{\alpha,\beta}(x)}{x})^{\frac{\alpha}{2}}.$$

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Some results concerning to polar derivative of polynomials

Abdullah Mir

Department of Mathematics, University of Kashmir, Srinagar, 190006, India.
Email:mabdullah_mir@yahoo.co.in

Abstract: Let $P(z)$ be a polynomial of degree n and for a complex number α, let

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

denote the polar derivative of the polynomial $P(z)$ with respect to α. In this paper, we shall establish some L^p -inequalities for the polar derivative of a polynomial with restricted zeros. Our result not only generalize some known polynomial inequalities but also a variety of interesting results can be deduced from these by a fairly uniform procedure.

Mathematics Subject Classifications (2010): 30A10, 30C10, 30C15.

Ky words and Phrases: Polar derivative, Polynomials, Zeros, L^p -inequalities in the complex domain.

1. Introduction

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. Then according to the well-known Bernstein's inequality [5] on the derivative of a polynomial, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin.

For the class of polynomials $P(z)$ having all zeros in $|z| \leq 1$, Turán [14] proved that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

Inequality (1.2) was refined by Aziz and Dawood [1] and they proved under the same hypothesis that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \{\max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)|\}. \tag{1.3}$$

Both the inequalities (1.2) and (1.3) are best possible and become equality for polynomials $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$. As an extension of (1.2), it was shown by Malik [13] that if $P(z)$ has all its zeros in $|z| \leq k, k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{1.4}$$

where as the corresponding extension of (1.3) and a refinement of (1.4) was given by Govil [10], who under the same hypothesis proved that

$$\max_{|z|=1}|P'(z)|\geq\frac{n}{1+k}\left\{\max_{|z|=1}|P(z)|+\frac{1}{k^{n-1}}\min_{|z|=k}|P(z)|\right\}. \tag{1.5}$$

In the literature, there already exist some refinements and generalizations of all the above inequalities, for example see Aziz and Shah [4], Dewan et al. [6], [8], Govil et al. [11] etc.

By considering the class of polynomials $P(z)=a_nz^n+\sum_{\upsilon=\mu}^{n}a_{n-\upsilon}z^{n-\upsilon}, 1\leq\mu\leq n$, of degree n having all zeros in $|z|\leq k, k\leq 1$, Aziz and Shah [4] (see also Dewan et al. [8]) proved

$$\max_{|z|=1}|P'(z)|\geq\frac{n}{1+k^{\mu}}\left\{\max_{|z|=1}|P(z)|+\frac{1}{k^{n-\mu}}\min_{|z|=k}|P(z)|\right\}. \tag{1.6}$$

For $\mu=1$, inequality (1.6) reduces to inequality (1.5).

Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to the point α. Then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\}=P'(z).$$

Aziz and Rather [3] extended (1.4) to the polar derivative of a polynomial and proved that if all the zeros of $P(z)$ lie in $|z|\leq k, k\leq 1$, then for every complex number α with $|\alpha|\geq k$,

$$\max_{|z|=1}|D_\alpha P(z)|\geq n\left(\frac{|\alpha|-k}{1+k}\right)\max_{|z|=1}|P(z)|. \tag{1.7}$$

for the class of polynomials $P(z)=a_nz^n+\sum_{\upsilon=\mu}^{n}a_{n-\upsilon}z^{n-\upsilon}, 1\leq\mu\leq n$, of degree n having all zeros in $|z|\leq k, k\leq 1$, Dewan et al. [7]) proved that if α is real or complex number with $|\alpha|\geq s_\mu$, then

$$\max_{|z|=1}|D_\alpha P(z)|\geq n\left(\frac{|\alpha|-s_\mu}{1+k^{\mu}}\right)\max_{|z|=1}|P(z)|, \tag{1.8}$$

where

$$s_\mu=\frac{n|a_n|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1}+\mu|a_{n-\mu}|}. \tag{1.9}$$

In the same paper Dewan et al. [7] proved for the same class of polynomials that if α is real or complex number with $|\alpha|\geq k^{\mu}$, then

$$\max_{|z|=1}|D_\alpha P(z)|\geq n(\frac{|\alpha|-k^\mu}{1+k^\mu})\max_{|z|=1}|P(z)|+n(\frac{|\alpha|+1}{k^{n-\mu}(1+k^\mu)})m$$
$$+n(\frac{k^\mu-A_\mu}{1+k^\mu})\max_{|z|=1}|P(z)|+n(\frac{A_\mu-k^\mu}{k^n(1+k^\mu)})m, \tag{1.10}$$

where $m=\min_{|z|=k}|P(z)|$ and

$$A_\mu=\frac{n(|a_n|-\frac{m}{k^n})k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n|-\frac{m}{k^n})k^{\mu-1}+\mu|a_{n-\mu}|}. \tag{1.11}$$

As a generalization of inequality (1.4), Dewan et al. [9] obtained an L^p inequality for the polar derivative of a polynomial and proved the following:

Theorem 1. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in* $|z|\leq k$ *, where* $k\leq 1$ *, then for every real or complex number* α *with* $|\alpha|\geq k$ *and for each* $r>0$ *,*

$$n(|\alpha|-k)\{\int_0^{2\pi}|P(e^{i\theta})|^r\,d\theta\}^{\frac{1}{r}}\leq\{\int_0^{2\pi}|1+ke^{i\theta}|^r\,d\theta\}^{\frac{1}{r}}\max_{|z|=1}|P'(z)|. \tag{1.12}$$

In the limiting case, when $r\to\infty$, the above inequality is sharp and equality holds for the polynomial $P(z)=(z-k)^n$ with $|\alpha|\geq k$.

The main aim of this paper is to provide an L^p-analogue of inequality (1.10) and to present a proof of it which is independent of Laguerre's theorem. Firstly, we shall prove the following generalization of inequality (1.8)

Theorem 2. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in* $|z|\leq k$ *, where* $k\leq 1$ *, then for every real or complex number* α *with* $|\alpha|\geq s_\mu$ *and for each* $r>0$, $p>1$, $q>1$ *with* $p^{-1}+q^{-1}=1$*, we have*

$$n(|\alpha|-s_\mu)\{\int_0^{2\pi}|P(e^{i\theta})|^r\,d\theta\}^{\frac{1}{r}}\leq\{\int_0^{2\pi}|1+k^\mu e^{i\theta}|^{pr}\,d\theta\}^{\frac{1}{pr}}\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr}\,d\theta\}^{\frac{1}{qr}}, \tag{1.13}$$

where s_μ is defined by the formula (1.9).

Remark 1. If we let $r\to\infty$ and $q\to\infty$ (so that $p\to 1$) in (1.13), we get inequality (1.8). If we divide both sides of (1.13) by $|\alpha|$ and let $|\alpha|\to\infty$, we get a result of Dewan et al. [[8], Theorem 1.3].

Next we shall prove the following more general result that as a special case provides a proof of inequality (1.10) which is independent of Laguerre's theorem.

Theorem 3. If $P(z)=a_nz^n+\sum_{\upsilon=\mu}^{n}a_{n-\upsilon}z^{n-\upsilon}, 1\leq\mu\leq n$, is a polynomial of degree n having all its zeros in $|z|\le k$, where $k\le 1$, then for every real or complex number α,β with $|\alpha|\ge k^{\mu}, |\beta|<1$ and for each $r>0$, $p>1$, $q>1$ with $p^{-1}+q^{-1}=1$, we have

$$n(|\alpha|-A_{\mu})\{\int_0^{2\pi}|P(e^{i\theta})-\frac{m\beta e^{in\theta}}{k^n}|^r\,d\theta\}^{\frac{1}{r}}$$

$$\le\{\int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}\,d\theta\}^{\frac{1}{pr}}\{\int_0^{2\pi}|D_{\alpha}P(e^{i\theta})-\frac{\alpha\beta mne^{i(n-1)\theta}}{k^n}|^{qr}\,d\theta\}^{\frac{1}{qr}}, \tag{1.14}$$

where $m=\min_{|z|=k}|P(z)|$ and A_{μ} is defined by the formula (1.11).

Remark 2. If we let $r\to\infty$ and $q\to\infty$ (so that $p\to 1$) in (1.14), we get

$$\max_{|z|=1}|D_{\alpha}P(z)-\frac{\alpha\beta mnz^{n-1}}{k^n}|$$

$$\ge n(\frac{|\alpha|-A_{\mu}}{1+k^{\mu}})\max_{|z|=1}|P(z)-\frac{m\beta z^n}{k^n}|. \tag{1.15}$$

Let z_0 be on $|z|=1$, such that $|P(z_0)|=\max_{|z|=1}|P(z)|$, then from (1.15), we get

$$|\{D_{\alpha}P(z)\}_{z=z_0}-\frac{\alpha\beta mnz_0^{n-1}}{k^n}|$$

$$\ge n(\frac{|\alpha|-A_{\mu}}{1+k^{\mu}})|P(z_0)-\frac{m\beta z^n}{k^n}|$$

$$\ge n(\frac{|\alpha|-A_{\mu}}{1+k^{\mu}})\{|P(z_0)|-\frac{m|\beta|z^n}{k^n}\}. \tag{1.16}$$

Since the polynomial $P(z)-\frac{m\beta z^n}{k^n}$ has all zeros in $|z|<k, k\le 1$ where $|\beta|<1$, therefore by Guass-Lucas theorem, the polynomial $P'(z)-\frac{mn\beta z^{n-1}}{k^n}$ also has all its zeros in $|z|<k, k\le 1$ and hence

$$|P'(z)|\ge\frac{mn|z|^{n-1}}{k^n},\ for\,|z|\ge k. \tag{1.17}$$

Because if (1.17) is not true, then there is a point $z=z_0$ with $|z_0|\ge k$ such that

$$|P'(z_0)|<\frac{mn|z_0|^{n-1}}{k^n}.$$

If we take $\beta=\frac{k^nP'(z_0)}{mnz_0^{n-1}}$, so that $|\beta|<1$, then with this choice of β, we have

$$|P'(z_0)|-\frac{mn\beta z_0^{n-1}}{k^n}=0,$$

where $|z_0|\geq k$, which contradicts the fact that all the zeros of $P'(z)-\dfrac{mn\beta z^{n-1}}{k^n}$ lie in $|z|<k, k\leq 1$.

Also for $|z|=1$,

$$\begin{aligned}|D_\alpha P(z)|&=|nP(z)+(\alpha-z)P'(z)|\\ &\geq|\alpha||P'(z)|-|nP(z)-zP'(z)|\\ &=|\alpha||P'(z)|-|Q'(z)|.\end{aligned}$$

Combining this inequality with Lemma 1, we get for $|z|=1$ and $|\alpha|\geq k^\mu$,

$$|D_\alpha P(z)|\geq(|\alpha|-k^\mu)|P'(z)|-\frac{mn}{k^{n-\mu}}. \qquad (1.18)$$

Inequality (1.18) in conjuction with (1.17) gives for $|z|=1$ and $|\alpha|\geq k^\mu$,

$$|D_\alpha P(z)|\geq\frac{|\alpha|mn}{k^n}. \qquad (1.19)$$

If in (1.15), we choose the arguement of β such that

$$\begin{aligned}&\left|\{D_\alpha P(z)\}_{z=z_0}-\frac{\alpha\beta mnz_0^{n-1}}{k^n}\right|\\ &=|\{D_\alpha P(z)\}_{z=z_0}|-\frac{mn|\alpha||\beta||z_0|^{n-1}}{k^n},\end{aligned}$$

which easily follows from (1.19), we obtain

$$|\{D_\alpha P(z)\}_{z=z_0}|-\frac{mn|\alpha||\beta||z_0|^{n-1}}{k^n}\geq n\left(\frac{|\alpha|-A_\mu}{1+k^\mu}\right)|P(z_0)|-n\left(\frac{|\alpha|-A_\mu}{1+k^\mu}\right)\frac{m|\beta|}{k^n}.$$

(1.20)

Since z_0 lie on $|z|=1$ and $|P(z_0)|=\max_{|z|=1}|P(z)|$, inequality (1.20) is equivalent to

$$|\{D_\alpha P(z)\}_{z=z_0}|\geq n\left(\frac{|\alpha|-A_\mu}{1+k^\mu}\right)\max_{|z|=1}|P(z)|-n\left(\frac{|\alpha|-A_\mu}{1+k^\mu}\right)\frac{m|\beta|}{k^n}+\frac{mn|\beta||\alpha|}{k^n}.$$

(1.21)

Now, if in (1.21) we make $|\beta|\to 1$, we get

$$\max_{|z|=1}|D_\alpha P(z)|\geq n\left(\frac{|\alpha|-A_\mu}{1+k^\mu}\right)\max_{|z|=1}|P(z)|+\frac{mn}{k^n}\left(\frac{|\alpha|k^\mu+A_\mu}{1+k^\mu}\right),$$

which is equivalent to (1.10) and this proves the required claim.

2. Lemmas

We need the following lemmas to prove the theorems.

Lemma 1. If $P(z)=a_n z^n+\sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}, 1\le\mu\le n$, is a polynomial of degree n having all its zeros in $|z|\le k, k\le 1$, and $Q(z)=z^n\overline{P(\frac{1}{\overline{z}})}$, then on $|z|=1$

$$|Q'(z)|\le k^{\mu}|P'(z)|, \tag{2.1}$$

where here and throughout this paper $Q(z)=z^n\overline{P(\frac{1}{\overline{z}})}$.

The above lemma is due to Aziz and Shah [2].

Lemma 2. If $P(z)=a_n z^n+\sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}, 1\le\mu\le n$, is a polynomial of degree n having all its zeros in $|z|\le k, k\le 1$, then on $|z|=1$

$$|Q'(z)|\le s_{\mu}|P'(z)|, \tag{2.2}$$

and

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right|\le k^{\mu},$$

where s_{μ} is defined by (1.9).

The above lemma is due to Aziz and Rather [2].

Lemma 3. If $P(z)=a_n z^n+\sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}, 1\le\mu\le n$, is a polynomial of degree n having all its zeros in $|z|\le k, k\le 1$, then on $|z|=1$

$$|Q'(z)|\le k^{\mu}|P'(z)|-\frac{nm}{k^{n-\mu}}, \tag{2.3}$$

where $m=\min_{|z|=k}|P(z)|$.

Lemma 4. If $P(z)=\sum_{\upsilon=0}^{n} a_{\upsilon} z^{\upsilon}$, is a polynomial of degree n having all its zeros in $|z|\le k, k>0$, then $|Q(z)|\ge\frac{m}{k^n}$ for $|z|\le\frac{1}{k}$ and in particular

$$|a_n|>\frac{m}{k^n}, \tag{2.4}$$

where $m=\min_{|z|=k}|P(z)|$.

Lemma 5. If $P(z)=a_nz^n+\sum_{\upsilon=\mu}^{n}a_{n-\upsilon}z^{n-\upsilon}, 1\le\mu\le n$, is a polynomial of degree n having all its zeros in $|z|\le k, k\le 1$,then

$A_\mu\le k^\mu$, (2.5)

where A_μ is defined by (1.11).

Lemma 6. The function

$$s_\mu(x)=\frac{nxk^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{nxk^{\mu-1}+\mu|a_{n-\mu}|},$$

where $k\le 1$ and $\mu\ge 1$, is a non-increasing function of x.

The above Lemmas 3, 4, 5 and 6 are due to Dewan et.al.[7].

3. Proofs of theorems

Proof of Theorem 2. Let $Q(z)=z^n\overline{P(\frac{1}{\overline{z}})}$, then $P(z)=z^n\overline{Q(\frac{1}{\overline{z}})}$ and it can be easily verified that for $|z|=1$,

$$|Q'(z)|=|nP(z)-zP'(z)| \tag{3.1}$$

and

$$|P'(z)|=|nQ(z)-zQ'(z)|. \tag{3.2}$$

Since $P(z)$ has all its zeros in $|z|\le k$, therefore, by using Lemma 1 and (3.2), we have for $|z|=1$,

$$|Q'(z)|\le k^\mu|nQ(z)-zQ'(z)|. \tag{3.3}$$

Now for every real or complex number α with $|\alpha|\ge s_\mu$, we have

$$\begin{aligned}|D_\alpha P(z)|&=|nP(z)-(\alpha-z)P'(z)|\\ &\ge|\alpha||P'(z)|-|nP(z)-zP'(z)|,\end{aligned}$$

which on using (3.1) and Lemma 2 gives for $|z|=1$,

$$\begin{aligned}|D_\alpha P(z)|&\ge|\alpha||P'(z)|-|Q'(z)|\\ &\ge(|\alpha|-s_\mu)|P'(z)|.\end{aligned} \tag{3.4}$$

Since $P(z)$ has all its zeros in $|z|\le k$, $k\le 1$, it follows by Guass-Lucas theorem that all the zeros of $P'(z)$ also lie in $|z|\le k$, $k\le 1$. This implies that the polynomial

$$z^{n-1}\overline{P'(\frac{1}{\overline{z}})}=nQ(z)-zQ'(z)$$

has all its zeros in $|z|\ge\frac{1}{k}\ge 1$. Therefore, it follows from (3.3) that the function

$$W(z)=\frac{zQ'(z)}{k^{\mu}(nQ(z)-zQ'(z))}$$

is analytic for $|z|\leq 1$ and $|W(z)|\leq 1$ for $|z|\leq 1$. Furthermore, $W(0)=0$ and so the function $1+k^{\mu}W(z)$ is subordinate to the function $1+k^{\mu}z$ for $|z|\leq 1$. Hence by a well-known property of sub-ordination [12], we have for each $r>0$,

$$\int_0^{2\pi}|1+k^{\mu}W(e^{i\theta})|^r\,d\theta\leq\int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^r\,d\theta. \tag{3.5}$$

Now

$$1+k^{\mu}W(z)=\frac{nQ(z)}{nQ(z)-zQ'(z)}$$

which gives with the help of (3.2) that for $|z|=1$

$$n|Q'(z)|=|1+k^{\mu}W(z)||P'(z)|. \tag{3.6}$$

Since $|P(z)|=|Q(z)|$ for $|z|=1$, therefore from (3.6), we get

$$|P'(z)|=\frac{n|P(z)|}{|1+k^{\mu}W(z)|},\ for\,|z|=1. \tag{3.7}$$

From (3.4) and (3.7), we deduce that for each $r>0$,

$$n^r(|\alpha|-s_{\mu})^r\int_0^{2\pi}|P(e^{i\theta})|^r\,d\theta\leq\int_0^{2\pi}|1+k^{\mu}W(e^{i\theta})|^r|D_{\alpha}P(e^{i\theta})|^r\,d\theta.$$

This gives with the help of Holder's inequality for $p>1, q>1$ with $p^{-1}+q^{-1}=1$, that

$$n^r(|\alpha|-s_{\mu})^r\int_0^{2\pi}|P(e^{i\theta})|^r\,d\theta\leq\{\int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}\,d\theta\}^{\frac{1}{p}}\{\int_0^{2\pi}|D_{\alpha}P(e^{i\theta})|^{qr}\,d\theta\}^{\frac{1}{q}}.$$

The above inequality in conjunction with (3.5) gives

$$n(|\alpha|-s_{\mu})\{\int_0^{2\pi}|P(e^{i\theta})|^r\,d\theta\}^{\frac{1}{r}}\leq\{\int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}\,d\theta\}^{\frac{1}{pr}}\{\int_0^{2\pi}|D_{\alpha}P(e^{i\theta})|^{qr}\,d\theta\}^{\frac{1}{qr}}.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. By hypothesis, the polynomial

$$P(z)=a_nz^n+\sum_{\upsilon=\mu}^{n}a_{n-\upsilon}z^{n-\upsilon}, 1\leq\mu\leq n$$

has all its zeros in $|z|\leq k$, $k\leq 1$. If $P(z)$ has a zero on $|z|=k$, then $m=0$ and the result follows from Theorem 2 in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z|<k$, $k\leq 1$ so that $m>0$.

Now $m \leq |P(z)|$ for $|z| = k$, therefore, if β is any real or complex number with $|\beta| < 1$, then

$$\left|\frac{m\beta z^n}{k^n}\right| < |P(z)| \ for\, |z| = k.$$

Since all the zeros of $P(z)$ lie in $|z| < k$, it follows by Rouche's theorem that all the zeros of $P(z) - \frac{m\beta z^n}{k^n}$ also lie in $|z| < k$, $k \leq 1$. Hence we can apply Theorem 2 to $P(z) - \frac{m\beta z^n}{k^n}$ and obtain for $|\alpha| \geq k^\mu \geq s'_\mu$, $r > 0$ and $p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n(|\alpha| - s_\mu)\left\{\int_0^{2\pi} \left|P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}\right|^r d\theta\right\}^{\frac{1}{r}}$$

$$\leq \left\{\int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}\right|^{qr} d\theta\right\}^{\frac{1}{qr}}, \tag{3.8}$$

where

$$s'_\mu = \frac{n\left|a_n - \frac{m\beta}{k^n}\right| k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n\left|a_n - \frac{m\beta}{k^n}\right| k^{\mu-1} + \mu |a_{n-\mu}|}. \tag{3.9}$$

Since for every β with $|\beta| < 1$, we have

$$\left|a_n - \frac{m\beta}{k^n}\right| \geq |a_n| - \frac{m|\beta|}{k^n} \geq |a_n| - \frac{m}{k^n} \tag{3.10}$$

and $|a_n| > \frac{m}{k^n}$ by Lemma 4. Now combining (3.9), (3.10) and Lemma 6 for every β with $|\beta| < 1$, we get

$$\begin{aligned} s'_\mu &= \frac{n\left|a_n - \frac{m\beta}{k^n}\right| k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n\left|a_n - \frac{m\beta}{k^n}\right| k^{\mu-1} + \mu |a_{n-\mu}|} \\ &\leq \frac{n(|a_n| - \frac{m\beta}{k^n})k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n(|a_n| - \frac{m\beta}{k^n})k^{\mu-1} + \mu |a_{n-\mu}|} \\ &= A_\mu. \end{aligned} \tag{3.11}$$

Since by Lemma 5, we have $A_\mu \leq k^\mu$, it follows from (3.8) and (3.11) that for every α

with $|\alpha|\ge k^{\mu}$, $r>0$ and $p>1$, $q>1$ with $p^{-1}+q^{-1}=1$, that

$$n(|\alpha|-A_{\mu})\{\int_{0}^{2\pi}|P(e^{i\theta})-\frac{m\beta e^{in\theta}}{k^{n}}|^{r}\,d\theta\}^{\frac{1}{r}}$$

$$\le\{\int_{0}^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}\,d\theta\}^{\frac{1}{pr}}\{\int_{0}^{2\pi}|D_{\alpha}P(e^{i\theta})-\frac{\alpha\beta mne^{i(n-1)\theta}}{k^{n}}|^{qr}\,d\theta\}^{\frac{1}{qr}},$$

which is inequality (3.14) and this completes the proof of Theorem 3.

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Convergence Rate Estimate of Wavelet Expansions in Sobolev Space

Neyaz Ahmad Sheikh and Owais Ahmad

Department of Mathematics, National Institute of Technology Srinagar (India)-190006.
Email: neyaznit@yahoo.co.in; siawoahmad@gmail.com.

Abstract: The main aim of this paper is to show the convergence of wavelet expansion to inverse Fourier transform and to zero when the scaling function is band-limited. Moreover, the estimate rate of the said convergence is obtained when the function f belongs to Sobolev space $H^s(R)$.

Keywords and Phrases: Convergence, Fourier transform, Scaling Function, Sobolev space.
AMS Subject Classification: 42C15, 42C40.

1. Introduction

Multiresolution approximations of $L^2(R)$ are the most natural frame work in the study of wavelet expansions [4]. let $\{V_m\}_{m\in Z}$ to denote a multiresolution analysis of $L^2(R)$ generated by a scaling function ϕ which is band-limited [3] and ψ to denote an orthogonal wavelet generated by ϕ, which gives rise to orthonormal basis $\{\psi_{m,n}\}_{m,n\in Z}$ $of L^2(R)$, where

$$\psi_{m,n}(x) = 2^{\frac{m}{2}}\psi(2^m x - n), \forall \; m,n \in Z.$$

Let

$$W_m = \overline{span}\{\psi_{m,n} \mid n \in Z\}.$$

and,

$$V_m = \overline{span}\{\phi_{m,n} \mid n \in Z\},$$

where,

$$\phi_{m,n}(x) = 2^{\frac{m}{2}}\phi(2^m x - n), \forall \; m,n \in Z,$$
$$V_m = V_{m-1} \oplus W_{m-1} = \ldots = \oplus_{k\le m-1} W_k.$$

Thus

$$L^2(R) = V_m \oplus (\oplus_{k\ge m} W_k) = \oplus_{m\in Z} W_m.$$

The translates of $\phi(x)$ are the orthonormal basis of V_0 while the translates of $\psi(x)$ are an orthogonal basis of W_0 [2]. The same holds for their dilations in V_j *and* W_j. Therefore, each $f \in L^2(R)$ has two following representations

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$$f = \sum_{n\in Z} a_{m,n}\phi_{m,n} + \sum_{k\geq m}\sum_{n\in Z} b_{k,n}\psi_{k,n,}) \qquad (1.11)$$

$$f = \sum_{k\in Z}\sum_{n\in Z} b_{k,n}\psi_{k,n}, \qquad (1.12)$$

where the convergence is taken in the sense of $L^2(R)$. The first term in (1.11) is said to be the partial sum of the wavelet expansions which is the projection of f onto the subspace V_m in the L^2 norm. The partial sum f_m of the wavelet expansion has two corresponding forms

$$f_m = \sum_{n\in Z} < f_m, \psi_{m,n} > \psi_{m,n} = \sum_{n\in Z} < f, \psi_{m,n} >, \qquad (1.13)$$

$$f_m = \sum_{k\leq m-1}\sum_{n\in Z} < f_k, \psi_{k,n} > \psi_{k,n} = \sum_{k\leq m-1}\sum_{n\in Z} < f, \psi_{k,n} > \psi_{k,n}, \qquad (1.14)$$

where the convergence is taken in the sense of $L^2(R)$.

In [11], Walter studied the convergence of f_m to f when f is continuous and the scale function satisfies

$$|\phi(x)| \leq \frac{C}{(1+|x|)^{1+\beta}}, \forall x \in R,$$

where $C, \beta > 0$ are constants. He established the pointwise and uniform convergence. Xiehua [11] gave the approximation degree of f_m to f. Further, the convergence of wavelet expansions of periodic functions is discussed in [1, 9] and the approximation degree of f_m to f was obtained. In [8,7], Gibbs phenomena for wavelet expansions was established. On the other hand, Mallat [4] gave a result which shows a relation between degree of convergence in L^2 - norm and the Sobolev space $H^s(R)$.

In the present paper, we have established the convergence of wavelet expansions to inverse Fourier transform and to zero when the scaling function is band-limited. Moreover, we established the estimate rate of the convergence in Sobolev space, $H^s(R)$.

2. Preliminaries

In the present section, we list some definitions

Let $f \in L^2(R)$ define the Fourier transform of f as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

The compact support of $\hat{f}$ is defined as $supp\hat{f} = clos\{\omega \,|\, \hat{f}(\omega) \neq 0\}$. If support of $\hat{f}$ is bounded, then f is called band-limited function. If $\hat{f} \in L^1(R)$, we define the inverse Fourier transform of $\hat{f}$ as

$$\hat{f}^{-1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega.$$

Let $H^s(R)$ denote the Sobolev space of order s [5], which is given by

$$H^s(R) = \{ f \in L^2(R) : | \hat{f}(\omega) | (1+\omega^2)^{\frac{s}{2}} \in L^2(R)\}.$$

For $f \in H^s(R)$, we define the norm

$$\| f \|_2^s = \sqrt{\{ \int_{-\infty}^{\infty} | \hat{f}(\omega) |^2 \ (1+\omega^2)^s d\omega\}}.$$

3. Main Results

Theorem 3.1. *Let supp* $\hat{\psi} \subset I = \{x : a \leq| x |\leq 2a, a > 0\}$. *If* $f \in L^2(R)$ *and* $\hat{f} \in L^1(R)$ *then*

(i) $f_m(x) \to \hat{f}^{-1}$ uniformly on R as m $\to +\infty$.

(ii) $f_m(x) \to 0$ uniformly on R as m $\to -\infty$.

Proof. Before we prove the theorem, we first state and prove the following lemma

Lemma 3.2. Assume Supp $\hat{\psi} \subset I = \{x : a \leq| x |\leq 2a, a > 0\}$. If $f \in L^2(R)$ and $\hat{f} \in L^1(R)$, then

$$\overline{\hat{\psi}(2^{-k}\omega)}\sum_{n \in Z}\psi(2^k x - n)e^{-i2^{-k}\omega n} = e^{i\omega x}, \forall \omega \in 2^k I, k = -\infty, \ldots, m-2, m-1$$

Proof of lemma 3.2. Let $H_k(x,\omega) = \overline{\hat{\psi}(2^{-k}\omega)}\sum_{n \in Z}\psi(2^k x - n)e^{-i2^{-k}\omega n}$,

Since for given x_o and integer k, there exists a continuous function $g \in L^1(R) \cap L^2(R)$ such that [6]

$$\hat{g}(\omega) = \overline{H_k(x_o,\omega) - e^{i\omega x_o}}, \omega \in 2^k I; \ \ \hat{g}(\omega) = 0 \ \ e.w$$

Also we have

$$g_k(x) = \frac{1}{2\pi}\int_{2^k I} \hat{g}(\omega)H_k(x,\omega)d\omega$$

As g is continuous, we have

$$g(x) = \frac{1}{2\pi}\int_{2^k I} \hat{g}(\omega)e^{i\omega x}d\omega$$

Here $g_k(x) \to g(x), as\ k \to +\infty$ .Therefore

$$\int_{2^k I} \hat{g}(\omega)H_k(x,\omega)d\omega = \int_{2^k I} \hat{g}(\omega)e^{i\omega x}d\omega,$$

which is equivalent to

$$\int_{2^k I} \hat{g}(\omega)[H_k(x,\omega)-e^{i\omega x}]d\omega = 0.$$

Choose $x = x_o$, we get

$$\int_{2^k I} [\overline{H_k(x_o,\omega)-e^{i\omega x_o}}][H_k(x_o,\omega)-e^{i\omega x_o}]d\omega = 0$$

implies,

$$\int_{2^k I} |H_k(x_o,\omega)-e^{i\omega x_o}|^2 \, d\omega = 0$$

implies,

$$|H_k(x_o,\omega)-e^{i\omega x_o}|^2 = 0$$

implies,

$$H_k(x_o,\omega) = e^{i\omega x_o}, \qquad \omega \in 2^k I$$

Since x_o is arbitrary. This completes the proof of the lemma.

Proof of Theorem 3.1. *For* $f \in L^2(R)$, *we have*

$$f_m(x) = \sum_{k=-\infty}^{m-1}\sum_{n\in Z} < f, \psi_{k,n}(x) > \psi_{k,n}(x)$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{m-1}\sum_{n\in Z}\int_{-\infty}^{+\infty} f(t)\overline{\psi_{k,n}(t)}dt\,\psi_{k,n}(x)$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{m-1}\sum_{n\in Z}\int_{-\infty}^{+\infty} f(t)\,\hat{f}(\omega)\overline{\hat{\psi}(2^{-k}\omega)}e^{-i2^{-k}\omega n}d\omega\,\psi(2^k x-n)$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{m-1}\sum_{n\in Z}\int_{2^k I} f(t)\,\hat{f}(\omega)\overline{\hat{\psi}(2^{-k}\omega)}e^{-i2^{-k}\omega n}d\omega\,\psi(2^k x-n)$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{m-1}\int_{2^k I} f(t)\,\hat{f}(\omega)\overline{\hat{\psi}(2^{-k}\omega)}\sum_{n\in Z}\psi(2^k x-n)e^{-i2^{-k}\omega n}d\omega.$$

By using lemma (3.2), the above statement can be written as,

$$f_m(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{m-1}\int_{2^k} \hat{f}(\omega)e^{i\omega x}d\omega$$

$$= \frac{1}{2\pi}\int_{-2^m a}^{2^m a} \hat{f}(\omega)e^{i\omega x}d\omega.$$

Now we have

$$| f_m(x) - \hat{f}^{-1}(x) | = | \frac{1}{2\pi} \int_{-2^m a}^{2^m a} \hat{f}(\omega) e^{i\omega x} d\omega - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega x} d\omega |$$

$$= | \frac{1}{2\pi} \int_{|\omega| \geq 2^m a} \hat{f}(\omega) e^{i\omega x} d\omega | .$$

Letting $m \to +\infty$, so conclusion (i) is established.
Also,

$$| f_m(x) | = | \frac{1}{2\pi} \int_{|\omega| \leq 2^m a} \hat{f}(\omega) e^{i\omega x} d\omega |$$

implies,

$$| f_m(x) | \leq \frac{1}{2\pi} \int_{|\omega| \leq 2^m a} | \hat{f}(\omega) e^{i\omega x} | d\omega.$$

Letting $m \to -\infty$, so conclusion (ii) holds good.
Thus the proof of theorem is completed.
Now the next theorem provides the estimate rate of the above established convergence in Sobolev space.

Theorem 3.3. *Let the scaling function* ϕ *be continuous, supp* $\hat{\phi} \subset I = [-a, a]$ *and* $|\hat{\phi}(\omega)| = 1$ *a.e on* $[-a, a]$. If $f \in L^2(R)$ and $\hat{f} \in L^1(R),$ then

(i) $f_m(x) \to \hat{f}^{-1}(x)$ uniformly on R as m $\to +\infty$. Further, if $f \in H^s(R)$ and $s > \frac{1}{2}$, then

$$| f_m(x) - \hat{f}^{-1} | \leq \frac{\| f \|_2^s}{\pi \sqrt{2s-1} a^{s-\frac{1}{2}}} 2^{m(s-\frac{1}{2})};$$

(ii) $f_m(x) \to 0$ uniformly on R as m $\to -\infty$. Further, if $f \in H^s(R)$ and $s < \frac{1}{2}$, then

$$| f_m(x) | \leq \frac{\| f \|_2^s}{\pi \sqrt{1-2s} a^{s-\frac{1}{2}}} 2^{m(s-\frac{1}{2})}.$$

Proof. By given hypothesis, ϕ is continuous, so each $\phi_{m,n}$ is continuous. Therefore, we have

$$\phi_{m,n}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\phi}_{m,n}(\omega) e^{i\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{-2^m a}^{2^m a} \hat{\phi}_{m,n}(\frac{\omega}{2^m}) \frac{e^{-i\frac{n}{2^m}\omega}}{2^{\frac{m}{2}}} e^{i\omega x} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2^m a}^{2^m a} \hat{\phi}_{m,n}(\frac{\omega}{2^m}) \frac{e^{-i\frac{n}{2^m}\omega}}{\sqrt{2^{m+1}\pi}} e^{i\omega x} d\omega.(3.31)$$

On the other hand, by parseval identity of Fourier transform we have

$$< f, \phi_{m,n} > = \frac{1}{2\pi} < \hat{f}, \hat{\phi}_{m,n} >$$

$$= \frac{1}{2\pi} \int_{-2^m a}^{2^m a} \hat{f}(\omega) \overline{\hat{\phi}_{m,n}(\omega)} d\omega$$

$$= \frac{1}{2\pi} \int_{-2^m a}^{2^m a} \hat{f}(\omega) \overline{\hat{\phi}(\frac{\omega}{2^m})} \frac{e^{i\frac{n}{2^m}\omega}}{2^{\frac{m}{2}}} d\omega$$

$$= \frac{1}{2\pi} \int_{-2^m a}^{2^m a} \hat{f}(\omega) \overline{\hat{\phi}(\frac{\omega}{2^m})} \frac{e^{i\frac{n}{2^m}\omega}}{\sqrt{2^{m+1}a}} d\omega.(3.32)$$

It is well known that the system $\{\frac{e^{i\frac{n}{2^m}\omega}}{\sqrt{2^{m+1}a}}\}_{n\in Z}$ is an orthonormal basis for $L^2[-2^m a, 2^m a]$. Therefore, from equations (3.31), (3.32) and the Bessel equality of Fourier series [12], we get

$$f_m(x) = \frac{1}{2\pi} \int_{-2^m a}^{2^m a} \hat{f}(\omega) e^{i\omega x} |\hat{\phi}(\frac{\omega}{2^m})|^2 \, d\omega.$$

Moreover, $|\hat{\phi}(\frac{\omega}{2^m})|^2 = 1 \, a.e \, on \, [-2^m \pi, 2^m \pi]$ .Therefore, we have

$$f_m(x) = \frac{1}{2\pi} \int_{-2^m a}^{2^m a} \hat{f}(\omega) e^{i\omega x} d\omega.$$

Now we have

$$| f_m(x) - \hat{f}^{-1}(x) | = | \frac{1}{2\pi} \int_{-2^m a}^{2^m a} \hat{f}(\omega) e^{i\omega x} d\omega - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega x} d\omega |$$

$$= | \frac{1}{2\pi} \int_{|\omega| \geq 2^m a} \hat{f}(\omega) e^{i\omega x} d\omega |$$

$$\leq \frac{1}{2\pi} \int_{|\omega| \geq 2^m a} |\hat{f}(\omega)| \, d\omega$$

Letting m $\to +\infty$, we get $f_m(x) \to \hat{f}^{-1}(x)$ uniformly on R.

On the otherhand, we have

$$|f_m(x) - 0| \leq \frac{1}{2\pi} \int_{|\omega| \leq 2^m a} \hat{f}(\omega) d\omega.$$

Letting $m \to -\infty$, we get $f_m(x) \to 0$ uniformly on R.

When $f \in H^s(R)$ and $s > \frac{1}{2}$, we have

$$|f_m(x) - \hat{f}^{-1}(x)| \leq \frac{1}{2\pi} \left| \int_{2^m a}^{+\infty} \hat{f}(\omega) e^{i\omega x} d\omega \right| + \left| \int_{-\infty}^{-2^m a} \hat{f}(\omega) e^{i\omega x} d\omega \right| = I_1 + I_2 (say).$$

For I_1, we have from cauchy inequality that

$$I_1 \leq \frac{1}{2\pi} \int_{2^m a}^{+\infty} |\hat{f}(\omega)| \, d\omega = \frac{1}{2\pi} \int_{2^m a}^{+\infty} \frac{|\hat{f}(\omega)| (1+\omega^2)^{\frac{s}{2}}}{(1+\omega^2)^{\frac{s}{2}}} d\omega$$

$$\leq \frac{\| f \|_2^s}{2\pi} \left(\int_{2^m a}^{+\infty} \frac{1}{(1+\omega^2)^s} d\omega \right)^{\frac{1}{2}}$$

$$\leq \frac{\| f \|_2^s}{2\pi} \left(\int_{2^m a}^{+\infty} \frac{1}{\omega^{2s}} d\omega \right)^{\frac{1}{2}}$$

$$= \frac{\| f \|_2^s \, 2^{m(\frac{1}{2}-s)}}{2\pi \sqrt{2s-1} a^{s-\frac{1}{2}}}.$$

Similarly,

$$I_2 \leq \frac{\| f \|_2^s \, 2^{m(\frac{1}{2}-s)}}{2\pi \sqrt{2s-1} a^{s-\frac{1}{2}}}.$$

Thus we have ,

$$|f_m(x) - \hat{f}^{-1}| \leq \frac{\| f \|_2^s \, 2^{m(\frac{1}{2}-s)}}{\pi \sqrt{2s-1} a^{s-\frac{1}{2}}}$$

When $f \in H^s(R)$ and $s > \frac{1}{2}$, By using Cauchy inequality we have

$$| f_m(x) | \le \frac{1}{2\pi} \int_{-2^m a}^{2^m a} | \hat{f}(\omega) | \, d\omega$$

$$\le \frac{\| f \|_2^s}{\pi} \left(\int_0^{2^m a} \frac{1}{(1+\omega^2)^s} d\omega \right)^{\frac{1}{2}}$$

$$\le \frac{\| f \|_2^s}{\pi} \left(\int_0^{2^m a} \frac{1}{\omega^{2s}} d\omega \right)^{\frac{1}{2}}$$

$$= \frac{\| f \|_2^s \, 2^{m(\frac{1}{2}-s)}}{\pi \sqrt{1-2s} \, a^{s-\frac{1}{2}}}.$$

This completes the proof of the theorem.

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From the Eisenhart problem to Ricci solitons in K$\ddot{a}$hler manifolds.

M. M. Praveena, C. S. Bagewadi and P. Somashekhara

Department of Mathematics, Kuvempu University,
Shankaraghatta - 577 451, Shimoga, Karnataka, INDIA.
e-mail: mmpraveenamaths@gmail.com; prof_bagewadi@yahoo.co.in; somumathrishi@gmail.com

Abstract: In this paper we obtain the conditions for the existence of Ricci soliton in non-flat real and complex space form by using Eisenhart problem. Also it is proved that if $(g, v. \lambda)$ is Ricci soliton then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the sign of scalar curvature.

Key Words: K$\ddot{a}$hler manifolds, Complex space forms, parallel second order covariant tensor field, Einstein space, Ricci soliton
2010 Mathematics Subject Classification: $53C21, 53C25, 53C35, 53C55$.

1. Introduction

Ricci flow is an excellent mathematical model introduced by R. S. Hamilon [7] in simplifying the structure of the manifolds. It is defined for Riemannain manifolds of any dimension. It is a process which deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smoothing out the regularity in the metric. It is given by

$$\frac{\delta g(t)}{\delta t} = -2Ric\big(g(t)\big),$$

where g is Riemannian metric dependent on time t and $Ric(g)$ is Ricci tensor.
Let $\phi_t : M \longrightarrow M, t \in R$ be a family of diffeomorphisms and $(\phi_t : t \in R)$ is a one parameter family of abelian group called flow. It generates a vector field X_P given by

$$X_P f = \frac{df(\phi_t(P))}{dt}, f \in C^\infty(M).$$

If Y is a vector field then $L_X Y = lim_{t\longrightarrow 0} \frac{\phi_t^* Y - Y}{t}$ is known as Lie derivative of Y with respect to X. Ricci solitons move under the Ricci flow under $\phi_t : M \longrightarrow M$ of the initial metric i.e., they are stationary points of the Ricci flow in space of metrics. If g_0 is a metric on the codomain then $g(t) = \phi_t^* g_0$ is the pullback of g_0, is a metric on the domain. Hence if $g_0$is a solution of the Ricci flow on the codomain subject to condition $L_V g_0 + 2Ricg_0 + 2\lambda g_0 = 0$ on the codomain then $g(t)$ is the solution of the Ricci flow on the domain subject to the condition $L_V g + 2Ricg + 2\lambda g = 0$ on the domain by [12] under suitable conditions. Here $g_0$and $g(t)$ are metrics which satisfy Ricci flow.
Thus the equation in general

$$L_V g + 2Ricg + 2\lambda g = 0, \tag{1.1}$$

is called Ricci soliton, where S is Ricci tensor of M, L_V denotes the Lie derivative operator along the vector field V and λ a real scalar. It is said to be expanding, shrinking and steady according as $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$ respectively.
Thus Ricci solitons are generalizations of Einstein manifolds and they are also called as quasi Einstein manifolds by theoretical physicists.

Eisenhart [6] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor then it is reducible. In 1925, Levy [10] obtained the necessary and sufficient conditions for the existence of such tensors. Since then, many others investigated the Eisenhart problem of finding symmetric and skew-symmetric parallel tensors on various spaces and obtained fruitful results. For instance, by giving a global approach based on the Ricci identity. Sharma [11] investigated Eisenhart problem on non-flat real and complex space forms, in 1989.

Using Eisenhart problem the authors Calin and Crasmareanu [3], Bagewadi and Ingalahalli [9, 1], Debnath and Bhattacharyya [5], have studied the existence of Ricci solitons in f-Kenmotsu manifolds, α-Sasakian, Lorentzian α- Sasakian, Trans-Sasakian and $(LCS)_n$ manifolds.

In this paper we study Ricci solitons of Kähler manifolds using Eisenhart problem. We use the following results

Theorem 1.1. *[11] A symmetric parallel second order covariant tensor h in a non-flat real space form of dimension $n > 2$ is a scalar multiple of the metric tensor i.e.,*

$$h(X,W) = \frac{tr(H)}{n} g(X,W) \tag{1.2}$$

where X, W are vector fields and H is associated $(1,1)$ tensor field.

Theorem 1.2. *[11] A parallel second order covariant tensor h in a non flat complex space form is a linear combination (with constant coefficients) of the underlying Kählerian metric and Kählerian 2-form i.e.,*

$$h(X,W) = \frac{1}{n}[(tr.H)g(X,W) + tr.(HJ)\Omega(X,W)] \tag{1.3}$$

where X, W are vector fields, J is complex structure tensor of type $(1,1)$, Ω is a Kählerian 2-form and H is a associated $(1,1)$ tensor field.

2. Preliminaries

A Kähler manifold is an n(even)-dimensional manifold, with a complex structure J and a positive definite metric g which satisfies the following conditions;

$$J^2(X) = -X, \; g(JX, JY) = g(X,Y) \text{ and } (\nabla_X J)Y = 0, \tag{2.1}$$

where ∇ means covariant derivative according to the Levi-Civita connection.

The formulae [2]

$$R(X,Y) = R(JX,JY) \tag{2.2}$$

$$S(X,Y) = S(JX,JY) \tag{2.3}$$

$$S(X,JY) + S(JX,Y) = 0, \tag{2.4}$$

are well known for a Kähler manifold.

Definition 2.1. A Riemannian manifold with constant sectional curvature c is called a real space form, and its curvature tensor satisfies the equation

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{2.5}$$

Models for these spaces are the Euclidean spaces (c = 0), the spheres (c > 0) and the hyperbolic spaces (c < 0).

Definition 2.2. A Kählerian manifold with constant holomorphic sectional curvature c is said to be a complex space form, and its curvature tensor is given by

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ]. \tag{2.6}$$

i.e., more precisely we can define a complex space form M as follows: M is a Kählerian manifold of constant holomorphic sectional curvature c, with its complex structure tensor $J : J^2 = -I$,

Kählerian metric $g: g(JX,JY) = g(X,Y)$, Kählerian 2-form: $\Omega(X,Y) = g(X,JY)$, and the Kählerian connection ∇: $\nabla J = 0$.
The models now are C^n, CP^n or CH^n , depending on $c = 0,\ c > 0\ or\ c < 0$.

3. Parallel symmetric second order covariant tensor and Ricci soliton in a non flat real space form

We write the following corollary using Theorem 1.1

Corollary 3.1. A locally Ricci symmetric $(\nabla S = 0)$ non flat real space form is an Einstein manifold.

Proof. Take $h = S$ in (1.2). If $H = Q$ then $tr.Q = r$. Therefore the equation (1.2) can be written as

$$S(X,W) = \frac{r}{n} g(X,W). \tag{3.1}$$

Remark 3.2. The following statements for non-flat real space form are equivalent.

1. Einstein
2. locally Ricci symmetric
3. Ricci semi-symmetric
4. Ricci pseudo-symmetric i.e., $R \cdot S = L_S Q(g,S)$ where L_S is some function

$U_S = \{x \in M : S \neq c\frac{r}{n} g\ at\ X\}$ provided $L_S \neq c.$

Proof. The statements $(1) \to (2) \to (3)$ and $(3) \to (4)$ are trivial. Now, we prove the statement $(4) \to (1)$ is true.

Here $R \cdot S = L_S Q(g,S)$ means

$$(R(X,Y) \cdot S(U,V)) = L_S Q(g,S)(U,V;X,Y),$$

which implies

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = L_S[S((X \wedge Y)U,V) + S(U,(X \wedge Y)V)]. \tag{3.2}$$

Using equations (2.5) in (3.2) and simplifying we get

$$(c - L_S)[g(Y,U)S(X,V) - g(X,U)S(Y,V) + g(Y,V)S(U,X) - g(X,V)S(U,Y)] = 0 \tag{3.3}$$

Putting $Y = U = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i\ (1 \le i \le n)$ we get after simplification that

$$(c - L_S)[-nS(X,V) + rg(X,V)] = 0. \tag{3.4}$$

If $(c - L_S) \neq 0,$ then (3.4) reduced to

$$S(X,V) = \frac{r}{n} g(X,V). \tag{3.5}$$

Therefore, we conclude the following.

Lemma 3.3. A Ricci pseudo-symmetric in a non flat real space form is an Einstein manifold if $L_S \neq c.$

Corollary 3.4. Suppose that in a non flat real space form, the (0,2) type field $L_V g + 2S$ is parallel where V is a given vector field. Then (g,V) yield a Ricci soliton. In particular, if the given a non flat real space form is Ricci semi-symmetric with $L_V g$ parallel, we have same conclusion.

Proof. Follows from Theorem 1.1 and corollary 3.1, we have

$$(L_V g+2S)(Y,Z)=\frac{2}{n}(divV+r)g(Y,Z)$$

Using equation (3.5) in (1.1), we get

$$(L_V g)(X,Y)+2\frac{r}{n}g(X,Y)+2\lambda g(X,Y)=0. \tag{3.6}$$

Putting $X=Y=e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i\,(1\le i\le n)$, we get after simplification that

$$(L_V g)(e_i,e_i)+2\frac{r}{n}g(e_i,e_i)+2\lambda g(e_i,e_i)=0. \tag{3.7}$$

This implies

$$divV+r+\lambda n=0. \tag{3.8}$$

If V is solenoidal then $divV=0$. Therefore the equation (3.8) can be reduced to

$$\lambda=\frac{-r}{n}. \tag{3.9}$$

For a non-flat real space form $r=cn(n-1)$. Hence from (3.9), we can state the following.

Corollary 3.5. Let (g,V,λ) be a Ricci soliton in a non flat real space form of dimension $(n>2)$. Then V is solenoidal if and only if it is shrinking i.e., spheres, steady i.e., Euclidean space and expanding i.e., hyperbolic spaces.

4. Parallel symmetric second order covariant tensor and Ricci soliton in a non flat complex space form

We write the following corollary using Theorem 1.2

Corollary 4. 1. A locally Ricci symmetric $(\nabla S=0)$ non-flat complex space form is an Einstein manifold.

Proof. Take $h=S$ in (1.3). If $H=Q$ then $tr.Q=r$ and $tr.QJ=0$ by virtue of (2.4). Hence from (1.3) can be write

$$S(Y,Z)=\frac{r}{n}g(Y,Z). \tag{4.1}$$

Remark 4.2. The following statements for non flat complex space form are equivalent.

1. Einstein
2. Locally Ricci symmetric
3. Ricci semi-symmetric
4. Ricci pseudo-symmetric i.e., $R\cdot S=L_S Q(g,S)$ where L_S is some function

$U_S=\{x\in M: S\neq c\frac{r}{n}g\,at\,x\}$ provided $L_S\neq c.$

Proof. The statements $(1) \to (2) \to (3)$ and $(3) \to (4)$ are trivial. Now, we prove the statement $(4) \to (1)$ is true.

Here $R \cdot S = L_S Q(g, S)$ means

$$(R(X,Y) \cdot S(U,V)) = L_S Q(g,S)(U,V;X,Y).$$

Which implies

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = L_S[S((X \wedge Y)U,V) + S(U,(X \wedge Y)V)]. \tag{4.2}$$

Using equations (2.6) in (4.2) and putting $Y = U = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i \; (1 \le i \le n)$ we get after simplification that

$$c[nS(X,V) - rg(X,V)] = L_S[nS(X,V) - rg(X,V)]. \tag{4.3}$$

The above equation can be written as,

$$[L_S - c][nS(X,V) - rg(X,V)] = 0. \tag{4.4}$$

If $L_S - c \neq 0,$ then (4.4) reduced to

$$S(X,V) = \frac{r}{n} g(X,V). \tag{4.5}$$

Therefore, we conclude the following.

Lemma 4.3. A Ricci pseudo-symmetric non flat complex space form is an Einstein manifold if $L_S \neq c.$

Corollary 4.4. Suppose that on a non flat complex space form, the (0,2) type field $L_V g + 2S$ is parallel where V is a given vector field. Then (g,V) yield a Ricci soliton if JV is solenoidal. In particular, if the given non flat complex space form is Ricci semi-symmetric with $L_V g$ parallel, we have same conclusion.

Proof. From Theorem 1.2 and corollary 4.4, we have $\lambda = -\frac{r}{n}$ as seen below:

$$\begin{aligned}(L_V g + 2S)(Y,Z) &= \frac{1}{n}[tr(L_V g + 2S)g(Y,Z) + tr.((L_V g + 2S)J)\Omega(Y,Z)] \\ &= \frac{1}{n}[2(divV + r)g(Y,Z) + 2(divJV)\Omega(Y,Z) + 2(tr.SJ)\Omega(Y,Z)],\end{aligned} \tag{4.6}$$

by virtue of (2.4) the above equation becomes

$$(L_V g + 2S)(Y,Z) = \frac{2}{n}[(divV + r)g(Y,Z) + (divJV)\Omega(Y,Z)]. \tag{4.7}$$

By definition (g,V,λ) yields Ricci soliton. If $divJV = 0$ then $divV = 0$ becouse $JV = iV$ i.e.,

$$(L_V g + 2S)(Y,Z) = \frac{2r}{n} g(Y,Z) = -2\lambda g(Y,Z). \tag{4.8}$$

Therefore $\lambda = -\frac{r}{n}$

Corollary 4.5. Let (g, V, λ) be a Ricci soliton in a non flat complex space form of dimension $(n > 2)$. Then V is solenoidal if and only if it is shrinking i.e., CP^n, steady i.e., C^n and expanding i.e., CH^n.

Proof.

$$(L_V g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0, \tag{4.9}$$

using equation (4.5) in (4.9) we get

$$(L_V g)(Y,Z) + 2\frac{r}{n} g(Y,Z) + 2\lambda g(Y,Z) = 0, \tag{4.10}$$

Putting $Y = Z = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i\ (1 \le i \le n)$, we get

$$(L_V g)(e_i, e_i) + 2\frac{r}{n} g(e_i, e_i) + 2\lambda g(e_i, e_i) = 0. \tag{4.11}$$

The above equation implies

$$divV + r + \lambda n = 0. \tag{4.12}$$

If V is solenoidal then $divV = 0$. Therefore the equation (4.12) can be reduced to

$$\lambda = \frac{-r}{n}.$$

for a non-flat complex space form $r = \frac{c}{4}(n-2)$. Hence from the above $\lambda = \frac{-c}{4n}(n-2).$

Conclusion.

From corollaries 3.2 and 4.5 we conclude the following:
According to corollary 3.2

$$(L_V g + 2S)(Y,Z) = \frac{2}{n}(divV + r) g(Y,Z)$$

and in this equation the solenoidal condition does not affect for the existence of Ricci soliton in non flat real space form, but according to corollary 4.5

$$(L_V g + 2S)(Y,Z) = \frac{2}{n}[(divV + r) g(Y,Z) + (divJV)\Omega(Y,Z) \tag{4.13}$$

and in this equation solenoidal condition affects for the existence of Ricci soliton in non flat complex space form, unless $divJV = 0.$

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Submanifolds of codimension r of an HGF-structure manifold

C.S. Prasad[1] **and Jaya Upreti**[2]

1 Department of Mathematics, D. B. S. (P. G.) College, Kanpur-208006 (India)
2 Departments of Mathematics, Kumaun University, S. S. J. Campus, Almora-263601 (India)

Abstract: In this paper we consider an HGF structure manifold and show that its submanifolds of codimension r admits a generalized Norder r-contact metric structure. Conditions for the integrability of such a structure are studied. A result connecting the curvature tensors of the manifold and of its submanifold is established.

1. Preliminaries

Let M^n be an $n-$dimensional differentiable manifold of differentiability class C^∞. Let there exists on M^n a nonzero tensor field ϕ of the type $(1,1)$ satisfying

$$\phi = -a^2 I \tag{1.1}$$

where $'a'$ is a complex number. Then $\{\phi\}$ is said to give to M^n a hyperbolic differentiable structure, briefly known as HGF-structure and the manifold M^n is called HGF-manifold. Suppose further that M^n be endowed with a metric tensor G, such that

$$G(\phi X^*, \phi Y^*) - a^2 G(X^*,\ Y^*) = 0 \tag{1.2}$$

for arbitrary vector fields X^* and Y^* on M^n, then $\{\phi, G\}$ is said to give to M^n a hyperbolic metric or hyperbolic Hermite structure [1].

Let $'\phi(X^*,\ Y^*)$ is the tensor field of type $(0,2)$ given by

$$'\Phi(X^*,\ Y^*) = G(\phi X^*,\ Y^*). \tag{1.3}$$

We can prove the following results easily

$$\begin{gathered} '\Phi(\phi X^*,\ Y^*) = -'\Phi(X^*,\ \phi Y^*) = -a^2 G(X^*,\ Y^*) \\ '\Phi(\phi X^*,\ Y^*) = a^2 G(X^*,\ Y^*) \\ '\Phi(X^*,\ Y^*) = '\Phi(Y^*,\ X^*) \end{gathered} \tag{1.4}$$

If $\widetilde{D}$ be the Riemannian connection on M^n then

$$\widetilde{D}_{X^*} Y^* - \widetilde{D}_{Y^*} X^* = [X^*,\ Y^*], \qquad \widetilde{D}_{X^*} G = 0. \tag{1.5}$$

Let $\widetilde{N}$ be the Nijenhuis tensor formed with ϕ; then

$$\widetilde{N}(X^*,\ Y^*) = [\phi X^*,\ \phi Y^*] - \phi[\phi X^*,\ Y^*] - \phi[X^*,\ \phi Y^*] + \phi^2[X^*,\ Y^*]. \tag{1.6}$$

In an HGF-structure manifold the structure tensor ϕ is parallel if

$$(\widetilde{D}_{X^*\phi})(Y^*) = 0. \tag{1.7}$$

Let M^{n-r} be a submanifold of codimension r of the HGF-structure manifold M^n. Let there exists a tensor field F of type $(1,1), r,\ C^\infty$-contravariant vector fields $\underset{x}{U},\ r\ C^\infty 1-$forms $\overset{x}{u}$, (r, some finite integer) on M^{n-r} satisfying [5]

$$F^2 = -a^2 I - \textstyle\sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U}. \tag{1.8}$$

Also

$$\overset{x}{u}\circ F + \textstyle\sum_{x=1}^{r} \ominus_x^y \overset{x}{u} = 0, \qquad F \underset{x}{U} + \textstyle\sum_{\mathrm{y}=1}^{\mathrm{r}} \ominus_x^y \underset{y}{U}$$

$$\overset{x}{u}\,(\underset{y}{U}) + \textstyle\sum_{z=1}^{r}\ominus_z^x\ominus_y^z \;=\; -a^2\delta_y^x \tag{1.9}$$

where $x, y, z \;=\; 1, 2, \ldots\ldots, r,\ \delta_y^x$ denotes the Kronecker delta and $\ominus_x^y$ are scalar fields. [4]
If the submanifold of M^{n-r} admits a Riemannian metric g satisfying

$$g(FX, FY) + a^2 g(X, Y) + \textstyle\sum_{x=1}^{r}\overset{x}{u}(X)\overset{x}{u}(Y) = \;0, \tag{1.10}$$

then we say that M^{n-r} admits a generalized Norder r –contact metric structure and the manifold M^{n-r} a generalized Norder r –contact metric structure manifold.

2. Submanifolds of co-dimension r

Theorem 1. *The submanifold* M^{n-r} *of co-dimension r of an HGF-structure manifold* M^n *admits a generalized Norder* r*-contact metric structure.*

Proof. Let M^{n-r} be the submanifold of co-dimension r of an HGF-structure manifold M^n. If B denotes the differential of the immersion $i\colon M^{n-r} \to M^n$, a vector field X in the tangent space of M^{n-r} determines a vector field BX in that of M^n. Let $\underset{x}{N},\ x = 1, 2, \ldots\ldots\ldots r$ be r mutually orthogonal fields of unit normal vectors defined on M^{n-r}. Thus we have

$$G(BX, BY) = g(X, Y), \qquad G\left(BX, \underset{x}{N}\right) = 0, \qquad G\left(\underset{x}{N}, \underset{y}{N}\right) = \delta_x^y. \tag{2.1}$$

The vector fields ϕBX and $\phi \underset{x}{N}$ can be expressed as

(a) $\phi BX = BFX - \sum_{x=1}^{r}\overset{x}{u}(X)\,\underset{x}{N}$ (2.2)

(b) $\phi \underset{x}{N} = -\mathrm{B}\,\underset{x}{U} + \sum_{y=1}^{r}\ominus_x^y\ \underset{y}{N},$

where F is a $(1, 1)$ tensor field, $\overset{x}{u}$, $1-$ forms and $\underset{x}{U}$ vector fields on the submanifold M^{n-r} $(x = 1, 2,\ \ldots\ldots\ldots\ldots, r)$.

Operating by ϕ on both the sides of $(2.2(a))$ and using the equations (1.1) and (2.2) we have

$$-a^2BX = BF^2X - \sum_{y=1}^{r}\overset{y}{u}\,(FX)\underset{y}{N} - \sum_{x=1}^{r}\overset{x}{u}\,(X)\left\{-B\underset{x}{U} + \sum_{y=1}^{r}\ominus_x^y\ \underset{y}{N}\right\}.$$

Comparing tangential and normal vectors we get

$$F^2 = \;-a^2I - \textstyle\sum_{x=1}^{r}\overset{x}{u}\otimes\underset{x}{U}, \qquad\qquad \overset{y}{u}\circ F + \textstyle\sum_{x=1}^{r}\ominus_x^y\ \overset{x}{u} = 0. \tag{2.3}$$

Operating by ϕ on both the sides of $(2.2(b))$ and using the equations (1.1) and (2.2) we obtain

$$-a^2\underset{x}{N} = -\left\{BF\underset{x}{U} - \sum_{z=1}^{r}\overset{z}{u}\left(\underset{x}{U}\right)\underset{z}{N}\right\} + \sum_{y=1}^{r}\ominus_x^y\left\{-B\underset{y}{U} + \sum_{z=1}^{r}\ominus_y^z\ \underset{z}{N}\right\}.$$

On comparing the tangential and normal vectors we have

$$F\underset{x}{U} + \textstyle\sum_{y=1}^{r}\ominus_x^y\ \underset{y}{U} = 0,\ \overset{z}{u}\left(\underset{x}{U}\right) + \textstyle\sum_{y=1}^{r}\ominus_y^z\ominus_x^y \;=\; -a^2\,\delta_x^z. \tag{2.4}$$

If g is the induced metric on M^{n-r}, then by virtue of the equations (1.1), (2.1) and (2.2) we have

$$g(FX, FY) + \;a^2g(X, Y) + \textstyle\sum_{x=1}^{r}\overset{x}{u}\,(X)\overset{x}{u}(Y) = \;0. \tag{2.5}$$

In view of the equations (2.3), (2.4) and (2.5) the theorem follows.

Theorem 2. *A totally geodesic submanifold* M^{n-r} *with a generalized Norder r-contact structure of a HGF-structure manifold is integrable.*

Proof. Let $\widetilde{D}$ is the Riemannian connection on M^n and D the induced connection on the submanifold M^{n-r}. Then the equations of Gauss and Weingarten can be expressed as

$$\widetilde{D}_{BX}BY = B\,D_XY + \textstyle\sum_{x=1}^{r}\overset{x}{\mathcal{H}}(X,Y)\underset{x}{N}, \tag{2.6}$$

$$\widetilde{D}_{BX}\underset{x}{N} = -B\,\overset{x}{H}(X) + \textstyle\sum_{y=1}^{r}\ominus_x^y\,\underset{y}{N} \tag{2.7}$$

where $\overset{x}{H}(X,Y)$ are the second fundamental forms and

$$\overset{x}{\mathcal{H}}(X,Y) = g\left(\overset{x}{H}(X),Y\right). \tag{2.8}$$

Suppose that in HGF-structure manifold M^n the structure tensor F is parallel. Hence we have
$(\widetilde{D}_{BX}\phi)(BY) = 0$ or equivalently $\widetilde{D}_{BX}\phi BY = \phi\widetilde{D}_{BX}BY.$
By virtue of the equations (2.2), (2.6) and (2.7) the last equation takes the form

$$D_{BX}\left\{BFY - \sum_{x=1}^{r}\overset{x}{u}(Y)\underset{x}{N}\right\} = \phi\left\{B\,D_XY + \sum_{x=1}^{r}\overset{x}{\mathcal{H}}(X,Y)\underset{x}{N}\right\}$$

or equivalently

$$B\,D_XFY + \sum_{x=1}^{r}\overset{x}{H}(X,FY)\underset{x}{N} - \sum_{x=1}^{r}\overset{x}{u}(Y)\left\{-B\,\overset{x}{H}(X) + \sum_{y=1}^{r}\ominus_x^y\,\underset{y}{N}\right\}$$

$$= B\,FD_XY - \sum_{x=1}^{r}\overset{x}{u}(D_XY)\underset{x}{N} + \sum_{x=1}^{r}\overset{x}{\mathcal{H}}(X,Y)\left\{-B\underset{x}{U} + \sum_{y=1}^{r}\ominus_x^y\,\underset{y}{N}\right\}.$$

Comparing tangential vectors from both the sides we have

$$D_XFY + \sum_{x=1}^{r}\overset{x}{u}(Y)\overset{x}{H}(X) = FD_XY - \sum_{x=1}^{r}\overset{x}{\mathcal{H}}(X,Y)\underset{x}{U}$$

or equivalently

$$(D_XF)(Y) + \textstyle\sum_{x=1}^{r}\left\{\overset{x}{u}(Y)\overset{x}{H}(X) + \overset{x}{\mathcal{H}}(X,Y)\underset{x}{U}\right\} = 0. \tag{2.9}$$

The Nijenhu is tensor $N(X,Y)$ for the submanifold M^{n-r} can be written as

$$N(X,Y) = (D_{FX}F)(Y) - (D_{FY}F)(X) + F(D_YF)(X) - F(D_XF)(Y). \tag{2.10}$$

The submanifold M^{n-r} be totally geodesic, the necessary and sufficient condition is that

$$\overset{x}{\mathcal{H}}(X,Y) = 0 \qquad (x = 1, 2, \dots\dots\dots, r).$$

Thus in view of the equations (2.8) and (2.9) it follows that $D_XF = 0$. Hence in consequence of (2.10) we have $N(X,Y) = 0$. Thus we have the proof of the theorem.

3. Curvature tensors

Suppose that W, X, Y, Z are arbitrary vector fields on an open set A in the neighbourhood of a point of the submanifold M^{n-r}. If $'\tilde{R}$ and $'R$ are the Riemannian Christoffel curvature tensors of M^n and M^{n-r}, respectively, we have

$$'\tilde{R}(BW,BX,BY,BZ) = {}'R(W,X,Y,Z) + \textstyle\sum_{x=1}^{r}\left\{\overset{x}{\mathcal{H}}(X,Z)\overset{x}{\mathcal{H}}(W,Y) - \overset{x}{\mathcal{H}}(X,Y)\overset{x}{\mathcal{H}}(W,Z)\right\}. \tag{3.1}$$

If the manifold M^n admits constant holomorphic sectional curvature k, we have

$$
\begin{aligned}
{}'\tilde{R}(BW,BX,BY,BZ)& \\
&= \frac{k}{4}[G(BW,BZ)G(BX,BY) - G(BX,BZ)G(BW,BY) \\
&+ {}'\Phi(BW,BZ)'\Phi(BX,BY) - {}'\Phi(BX,BZ)'\Phi(BW,BY) \\
&+ 2\,{}'\Phi(BW,BX)\,{}'\Phi(BY,BZ)]
\end{aligned}
\tag{3.2}
$$

From equations (1.3) and (2.2) it can be proved that

$$
{}'\Phi(BX,BY) = {}'F(X,Y) \overset{\text{def}}{=} g(FX,Y). \tag{3.3}
$$

Hence, in view of equations (2.1) , (3.2) and (3.3) , the equation (3.2) takes the form

$$
\begin{aligned}
{}'R(W,X,Y,Z) = {} & \frac{k}{4}[g(W,Z)g(X,Y) - g(X,Z)g(W,Y) + {}'F(W,Z)'F(X,Y) - \\
& F(X,Z)'F(W,Y) + 2\,{}'F(W,X)'F(Y,Z)] + \\
& \textstyle\sum_{x=1}^{r} \left\{ \overset{x}{\mathcal{H}}(X,Y)\overset{x}{\mathcal{H}}(W,Z) - \overset{x}{\mathcal{H}}(X,Z)\overset{x}{\mathcal{H}}(W,Y) \right\}.
\end{aligned}
\tag{3.4}
$$

Thus we have

Theorem 3. *Let M^n be an HGC-structure manifold of constant holomorphic sectional curvature k. Then the curvature tensor of the submanifold M^{n-r} satisfies the equation* (3.4).

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Hybrid Projective Synchronization of incommensurate fractional order chaotic systems

Khan Ayub and Muzaffar Ahmad Bhat

Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India
Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India
E-mail : akhan12@jmi.ac.in; mzfar012@gmail.com

Abstract: This article deals with the hybrid projective synchronization between two identical incommensurate fractional order Lorenz system using active control technique. The chaotic attractors of the system are found for different fractional order time derivatives described in Caputo sense. Numerical simulation results which are carried out by using Adams-Boshforth-Moulton method show that the method is reliable and effective for hybrid projective synchronization of nonlinear dynamical evolutionary systems.

Keywords: fractional order systems; incommensurate systems; chaotic attractors; hybrid projective synchronization.
Mathematics Subject Classification (2010):37B25, 37D45, 37N30, 37N35, 70K99.

1. Introduction

The theory of derivatives of fractional order, i.e., non-integer order, goes back to Leibniz's note in his list to L'Hopital, dated 30 September 1695, in which the meaning of derivative of order one half was discussed. Fractional calculus is a 300 year old mathematical topic. Although it has a long history, the applications of fractional calculus to physics and engineering are just a recent focus of interest [1] and [2]. It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [3], dielectric polarization [4], electrode–-electrolyte polarization [5], electromagnetic waves [6], and quantum evolution of complex systems [7]. In recent years, chaotic phenomenon have been found in many fractional-order nonlinear systems, such as the fractional-order Lorenz chaotic system [8],[9], Chua's fractional-order chaotic circuit system [9], the fractional-order modified Duffing chaotic system [10], the fractional-order Rossler chaotic system [11],[12], the fractional-order Chen chaotic system [9]-[11], the fractional-order memristor chaotic system [13], and so on.

In 1999, projective synchronization was first proposed by Mainieri and Rehacek [14], where the drive and response systems were synchronized up to a scaling factor. Its proportional feature can be used to extend binary digital to M-nary digital communication for achieving fast communication [15]. Both complete synchronization and anti-phase synchronization are special cases of projective synchronization. Recently, various kinds of projective synchronization for fractional order chaotic systems without time-delay have been studied, such as hybrid projective synchronization [16], generalized projective synchronization [17], function projective synchronization [18], lag projective synchronization [19] and modified projective synchronization [20]

The study of hybrid projective synchronization between two identical incommensurate fractional order chaotic systems has invoked the interest in the authors to investigate the time required for hybrid projective synchronization between fractional order chaotic systems.Active control method, proposed by Bai and Lonngren [21] is simple and easy to implement in practical applications of synchronization of coupling of a pair of systems and has received huge attention during last few years [22, 23, 24]. Hybrid projective synchronization between identical and non-identical chaotic systems using active control method in both standard order and fractional-order systems are already been studied [25, 26, 27]. Keeping in view, the challenges for detecting transformation of dynamical variables between the identical or non-identical systems during synchronization tremendous applications of fractional calculus in various areas of science and engineering and effectiveness of the active control method, the authors are motivated to make an attempt to do a coupling of identical incommensurate fractional-order chaotic systems to receive different types of information from the systems due to its memory effect and greater degrees of flexibilities.

In this article, we have studied the hybrid projective synchronization between identical incommensurate fractional-order chaotic systems using active control method. Using the Adams–Boshforth–Moulton method [28, 29], computer simulations are carried out for different order fractional time derivatives and are displayed graphically to demonstrate the efficiency of the proposed approach. The authors think that the article will be a useful contribution to the scientific literature on the methods of control for nonlinear dynamical systems.

2. The Review and The Approximation of A Fractional Operator

The differintegral operator, denoted by ${}_aD_t^q$, is a combined differentiation-integration operator commonly used in fractional calculus. This operator is a notation for taking both the fractional derivative and the fractional integral in a single expression and is defined by:

$$ {}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q}, & : q > 0 \\ 1, & : q = 0 \\ \int_a^t (d\tau)^{-q}, & : q < 0 \end{cases} \tag{1} $$

There are some definitions for fractional derivatives[1]. The commonly used definitions are Grunwald-Letnikov, Riemann-Liouville, and Caputo definitions. The Grunwald-Letnikov definition is given by:

$$ \begin{aligned} {}_aD_t^q f(t) &= \frac{d^q f(t)}{d(t-a)^q} \\ &= \lim[\frac{t-a}{N}]^{-q} \sum_{j=0}^{N-1} (-1)^j \binom{q}{j} f(t - j[\frac{t-a}{N}]) \end{aligned} \tag{2} $$

The Riemann–-Liouville definition is the simplest and easiest definition to use. This definition is given by:

$$ \begin{aligned} {}_aD_t^q f(t) &= \frac{d^q f(t)}{d(t-a)^q} \\ &= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-q-1} f(\tau) d(\tau), \end{aligned} \tag{3} $$

where n is the first integer which is not less than q, i.e., $n-1 \le q < n$ and Γ is the Gamma function defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{4}$$

For functions $f(t)$ having n continuous derivatives for $t \ge 0$ where $n-1 \le q < n$, the Grunwald–Letnikov and the Riemann–Liouville definitions are equivalent. The Laplace transforms of the Riemann–Liouville fractional integral and derivative are given as follows:

$$L\{{}_0D_t^q f(t)\} = S^q F(s) : q \le 0. \tag{5}$$

$$L\{{}_0D_t^q f(t)\} = S^q F(s) - \sum_{k=0}^{n-1} S_0^k D_t^q f(0).n-1 < q \le n \in N \tag{6}$$

Unfortunately, the Riemann-Liouville fractional order derivative appears unsuitable to be treated by the Laplace transform technique in that it requires the knowledge of the non-integer order derivatives of the function at $t = 0$. This difficulty does not arise in the Caputo definition that is sometimes referred as smooth fractional derivative in literature. This definition of derivative is defined as

$${}_0D_t^q = \begin{cases} \dfrac{1}{\Gamma(m-q)} \displaystyle\int_0^t \dfrac{f^m(\tau)}{(t-\tau)^{q+1-m}} d\tau, & : m-1 < q < m \\ \dfrac{d^m f(t)}{dt^m}, & : q = m \end{cases} \tag{7}$$

where m is the first integer larger than q. It is found that the equations with Riemann–Liouville operators are equivalent to those with Caputo operators by homogeneous initial conditions assumption [1].

3. Stability of Fractional Order Systems

Stability of fractional order systems has been thoroughly investigated where necessary and sufficient conditions have been derived in [30]. The stability region of a linear set of fractional order equations, each of order q, such that $1 < q < 2$ is shown in figure 1. An autonomous system is asymptotically stable iff $|arg\lambda| > \frac{q\pi}{2}$ is satisfied for all eigenvalues λ of matrix A Also this system is stable iff $|arg\lambda| \ge \frac{q\pi}{2}$ is satisfied for all eigenvalues of a matrix A and those critical eigenvalues which satisfy $|arg\lambda| > \frac{q\pi}{2}$, and have geometric multiplicity one [30].

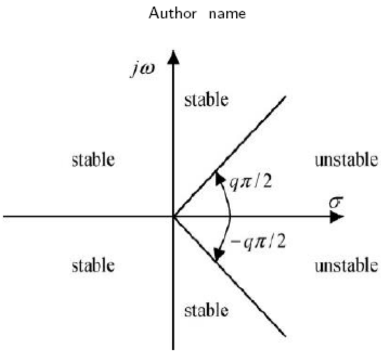

Figure 1 : Stability of fractional order systems such that $0 < q < 1$

4. System Description

The fractional order incommensurate Lorenz system is described as

$$\begin{aligned} \frac{d^{\alpha_1}x_1}{dt^{\alpha}} &= a_1(x_2 - x_1) \\ \frac{d^{\alpha_2}x_2}{dt^{\alpha}} &= c_1x_1 - x_1x_3 - x_2 \\ \frac{d^{\alpha_3}u_3}{dt^{\alpha}} &= x_1x_2 - b_1x_3 \end{aligned} \tag{8}$$

where α_1, α_2 and α_3 are the fractional orders and for the parameter values $a = 10, b = 8/3, c = 28$ the system yields a chaotic attractor. The chaotic attractors for different values of $\alpha_1 = 0.98, \alpha_2 = 0.99$ and $\alpha_3 = 1$ are displayed in Figs. 2(a-d)

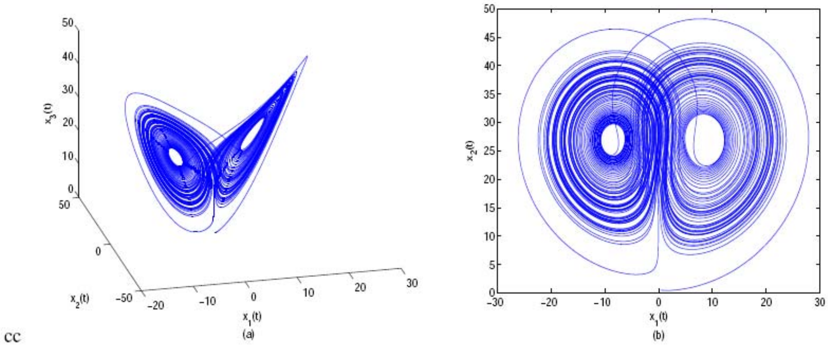

Figure 2: (a)3-Dimensional chaotic attractor of system (8) for $\alpha_1 = 0.98, \alpha_2 = 0.99$ and $\alpha_3 = 1$ (b) 2-Dimensional chaotic attractor of system (8) with $\alpha_1 = 0.98, \alpha_2 = 0.99$ and $\alpha_3 = 1$

Definition. The fractional order chaotic drive and response systems can be described as follows:

$$\frac{d^{\alpha_i}(x)}{dt^{\alpha_i}} = f(x);\ i = 1,2,3 \tag{9}$$

and

$$\frac{d^{\alpha_i}(y)}{dt^{\alpha}} = g(y) + \phi(x,y) \tag{10}$$

where $x \in \mathrm{R}^n, y \in \mathrm{R}^m$ are state vectors of the drive system (9), and the response system (10) and $f, g : R^n \to R^n$ are continuous vector functions, respectively, $\phi(x, y)$ is a vector controller to be designed.

For the drive system (9) and the response system (10), the Hybrid Projective Synchronization (HPS) is achieved if there exist an $n \times n$ invertible matrix A such that

$$\lim_{t\to\infty} \| e(t) \| = \| Ay - x \| = 0$$

where $|| \bullet ||$ is an Euclidean norm.

Remark. If $A = \sigma I; \sigma \in \mathrm{R}$, the hybrid projective synchronization problem will reduce to Projective Synchronization(PS) where I is an $n \times n$ matrix with proper dimensions.In particular if $\sigma = 1$ and $\sigma = -1$ the problem is further simplified to complete synchronization and anti-Phase synchronization,respectively. If $A = diag(a_1, a_2, ..., a_n)$ where $a_1, a_2, ..., a_n$ are not all zeros and $a_i \neq a_j$ for some i and j, then the Modified Projective Synchronization will appear. Therefore Complete synchronization, anti-synchronization, projective synchronization, and modified projective synchronization are the special cases of Hybrid Projective Synchronization.

5. Hybrid projective synchronization between incommensurate fractional order Lorenz system

In this section, the incommensurate fractional order Lorenz system is considered as the drive system

$$\begin{aligned}
\frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} &= a_1(x_2 - x_1) \\
\frac{d^{\alpha_2} x_2}{dt^{\alpha_2}} &= c_1 x_1 - x_1 x_3 - x_2 \\
\frac{d^{\alpha_3} x_3}{dt^{\alpha_3}} &= x_1 x_2 - b_1 x_3
\end{aligned} \tag{11}$$

and also the analogous response system as

$$\begin{aligned}\frac{d^{\alpha_1}y_1}{dt^{\alpha_1}} &= a_1(y_2-y_1)+u_1(t)\\ \frac{d^{\alpha_2}y_2}{dt^{\alpha_2}} &= c_1y_1-y_1y_3-y_2+u_2(t)\\ \frac{d^{\alpha_3}y_3}{dt^{\alpha_3}} &= y_1y_2-b_1y_3+u_3(t)\end{aligned} \tag{12}$$

where $u(t)=[u_1(t),u_2(t),u_3(t)]^T$ is the controller to be designed. To investigate the the hybrid projective synchronization of the system (11) and (12), we define the error states as
$e_1(t)=y_1(t)+2x_1(t),\ e_2(t)=y_2(t)-2x_2(t),\ e_3(t)=y_3(t)+x_3(t).$
The corresponding error dynamics is obtained by using (11) and (12) as

$$\begin{aligned}\frac{d^{\alpha_1}e_1}{dt^{\alpha_1}} &= a_1e_2-a_1e_1+4a_1x_1+u_1(t)\\ \frac{d^{\alpha_2}e_2}{dt^{\alpha_2}} &= c_1e_1-e_2-4x_2-y_1y_3-2x_1x_3+u_2(t)\\ \frac{d^{\alpha_3}e_3}{dt^{\alpha_3}} &= -b1e_3+y_1y_2+x_1x_2+u_3(t).\end{aligned} \tag{13}$$

Choosing the control functions as

$$\begin{aligned}u_1(t) &= a_1e_2-a_1e_1+4a_1x_1+v_1(t)\\ u_2(t) &= c_1e_1-e_2-4x_2-y_1y_3-2x_1x_3+v_2(t)\\ u_3(t) &= -b1e_3+y_1y_2+x_1x_2+v_3(t).\end{aligned} \tag{14}$$

By using the (14) into (13), we find that

$$\begin{aligned}\frac{d^{\alpha_1}e_1}{dt^{\alpha_1}} &= a_1e_2-a_1e_1+v_1(t)\\ \frac{d^{\alpha_1}e_2}{dt^{\alpha_2}} &= c_1e_1-e_2+v_2(t)\\ \frac{d^{\alpha_1}e_3}{dt^{\alpha_3}} &= -b1e_3+v_3(t).\end{aligned} \tag{15}$$

where $v_1(t),v_2(t),v_3(t)$ are the linear control inputs chosen such that the system (15) becomes stable.

Next consider

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ \end{pmatrix} = M \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ \end{pmatrix} \tag{16}$$

where M is a 3×3 matrix. In order to make the closed loop stable the matrix M is chosen in such a way that the eigenvalues λ_i of the matrix M of the feedback system satisfy $|\, arg(\lambda_i)\,| > \frac{\pi\alpha}{2}, i = 1,2,3.$

Consider the following matrix

$$M = \begin{pmatrix} a-1 & -a & 0 \\ -c & 0 & 0 \\ 0 & 0 & b-1 \\ \end{pmatrix}$$

Then the error dynamics becomes

$$D^{\alpha_i} e_i(t) = e_i(y) \tag{17}$$

Here all the eigenvalues of the the matrix M are -1, which satisfy $|\, arg(\lambda_i)\,| > \frac{\pi\alpha}{2}, i = 1,2,3$ $0 < \alpha < 1$. Therefor the error system converges to zero as $t \to \infty$ and therefore hybrid projective synchronization between the systems (11) and (12) is achieved.

5.1. Numerical simulations and results

During the simulation to demonstrate the hybrid projective synchronization behaviour between two identical incommensurate fractional order Lorenz chaotic system, the parameter values are considered as $a = 10, b = 8/3, c = 28,$ the time step size is taken as 0.005. The initial valves of the state variables $(x_1(0), x_2(0), x_3(0))$ and $(y_1(0), y_2(0), y_3(0))$ are taken as $(0.3, 0.4, 0.5)$ and $(0.1, 0.4, 0.7)$ respectively, so that the initial errors are $(0.7, -0.4, 1.2)$. Also we chose the scaling matrix $a = diag.(-2, 2, -1)$. Figure 3(a-d) respectively displays that the state vectors are synchronized in a hybrid projective synchronizing manner for the fractional-order derivatives $\alpha_1 = 0.98, \alpha_2 = 0.99, \alpha_3 = 1$

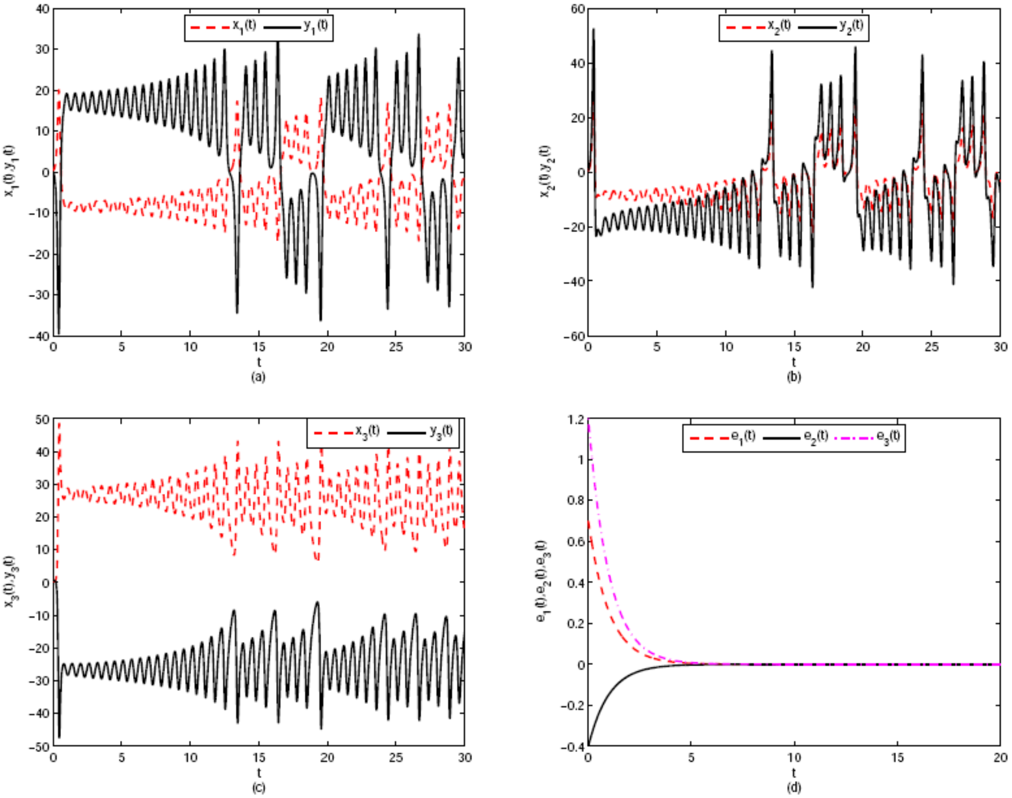

Figure 3 : (a)Response for states $x_1(t)$ and $y_1(t)$ (b) Response for states $x_2(t)$ and $y_2(t)$ (c) Response for states $x_3(t)$ and $y_3(t)$ (d) Synchronization errors $e_1, e_2 \& e_3$ between the system (11) and (12)

6. Conclusion

In this paper, the hybrid projective synchronization between identical commensurate fractional order system using active control using active control method based on fractional order stability theory have been investigated. Hybrid projective synchronization (HPS) is a more general definition of projective synchronization, in which the drive system and response system could be synchronized up to a vector function factor. Hybrid projective synchronization is different from the projective synchronization and more beneficial to enhance security of communication than any other synchronization because it is obvious that the unpredictability of the vector function factor in hybrid projective synchronization is more than that of the same scaling factor in PS. The numerical simulations exhibit the validity and feasibility of the proposed scheme.Numerical and computational result are in excellent agreement.

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On Special P^v-Symmetric Finsler Spaces

S.K. Tiwari

Department of Mathematics, K.S. Saket P.G. College, Ayodhya-Faizabad-224123 (India)
E-mail: sktiwarisaket@yahoo.com

Abstract: The object of present paper is to investigate the condition under which n-dimensional Finslerspace reduces to a p^V-symmetric Finsler space. We have obtained that certain special Finsler spaces becomes P^V-symmetric Finsler space under usual conditions. Some basic properties of aforesaid symmetric Finsler space have been discussed.

1. Introduction

Let F_n be an n-dimensional Finsler space with a fundamental function L(x,y), $(y^i = \dot{x}^i)$. From L(x,y), the following well known tensors are obtained :

$$l_i = \frac{\partial L}{\partial y^i}, h_{ij} = L\left(\frac{\partial l_i}{\partial y^i}\right). \tag{1.1}$$

These are called respectively the normalized supporting element and the angular metric tensor.

$$g_{ij} = \frac{1}{2}\left(\frac{\partial^2 L^2}{\partial y^i \partial y^j}\right), P_{ijk} = g_{ir}P^r_{jk}. \tag{1.2}$$

The fundamental tensor g_{ij} is also written as $g_{ij} = h_{ij} + l_i l_j$, P_{ijk} is called the V(hv) torsion tensor and satisfies $P_{ijk}y^k = 0$.

The V-covariant derivative $T^i_j\big|_k$ of Finsler tensor field of (1,1) type is defined by

$$T^i_j\big|_k = \frac{\partial T^i_j}{\partial y^k} + T^r_j C^i_{rk} - T^i_r C^r_{jk} \tag{1.3}$$

Definition 1.1. A Finsler space F_n, is called C^h-recurrent, if the (h) *hv*-torsion tensor C^i_{jk} satisfies the equation [2]

$$C^i_{jk|h} = K_h C^i_{jk}, C_{ijk|h} = K_h C_{ijk}, \tag{1.4}$$

where |h denotes the *h*-covariant differentiation, $K_h = K_h(x,y)$ is a co-variant vector field.
The equation (1.4) is denoted by the relations

(a) $C_{ijk}|_0 = K_0 C_{ijk}$ (1.5)
(b) $P_{ijk} = K_0 C_{ijk}$
(c) $P_i = K_0 C_i = C_i|_0$
(d) $P = K_0 C = C|_0$.

Definition 1.2. A Finsler space F_n, (n>2) is called a P-reducible, if the V(*hv*) torsion tensor P_{ijk} is of the form [1]

$$P_{ijk} = \frac{1}{n+1}\left(P_i h_{jk} + P_j h_{ki} + P_k h_{ij}\right). \tag{1.6}$$

Definition 1.3. A Finsler space F_n, (n>2) is called a C-reducible, if the *h*(*hv*)-torsion tensor C_{ijk} is of the following form [3]

$$C_{ijk} = \frac{1}{n+1}\left(C_i h_{jk} + C_j h_{ki} + C_k h_{ij}\right) \tag{1.7}$$

We shall use the following lemma [6].

Lemma. There exists a scalar α in a C-reducible Finsler space such that

$$LC_i\big|_j + C_i l_j + C_j l_i = \alpha h_{ij}$$

This equation satisfies in the following form

$$LP_i\big|_j + P_i l_j + P_j l_i = \alpha\big|_0\, h_{ij} \tag{1.8}$$

We list up miscellaneous formulas [5], which have been used in the chapter.

(a) $\dot{\partial}_i L = l_i = \frac{Y^i}{L}$ where $\partial_i = \frac{\partial}{\partial y^i}$ (1.9)

(b) $l_i = g^{ij} l_j = \frac{Y^i}{L}$

(c) $\dot{\partial}_j l^i = \frac{h^i_j}{L} = \frac{\left(\delta^i_j - l^i l_j\right)}{L}$

(d) $\dot{\partial}_j l^i = \frac{h^i_j}{L}$

(e) $\dot{\partial}_k h^i_j = -\left(h^i_k + h_{jk} l^i\right)$ and

(f) $\dot{\partial}_k h_{ij} = 2C_{ijk} - \left(h_{ik} l_j + h_{jk} l_i\right)/L$

Here, we shall list up the h-covariant and V-covariant derivatives of some fundamental tensors for latter use (6).

(a) $L_{|i} = 0$

$L\big|_i = l_i$

(b) $y^i{}_{|j} = 0$,

$y^i\big|_j = \delta^i_j$

(c) $l_{i|j} = 0$,

$l_{i|j} = \frac{1}{L} h_{ij}$, (1.10)

(d) $h_{ij|k} = 0$,

$h_{ij|k} = -\frac{1}{L}\left(h_{ik} l_j + h_{jk} l_i\right)$

(e) $g_{ij|k} = 0$,

$g^{ij}_{|k} = 0$, $g_{ij}\big|_k = 0$, $g^{ij}\big|_k = 0$

(f) $l^i|_j = 0$,

$$l^i\Big|_j = \frac{1}{L}\left(\delta^i_j - l^i l_j\right)$$

The V-curvature tensor S_{hijk} is given by

$$S_{hijk} = C_{hkr}C^r_{ij} + C_{hkr}P^r_{ij} - C_{hjr}C^r_{ik}. \tag{1.11}$$

This equation can be written as the using (1.5), we have

$$S_{hijk|0} = K_0 S_{hijk} = P_{hkr}C^r_{ij} + C_{hkr}P^r_{ij} - P_{hjr}C^r_{ik} - C_{hjr}P^r_{ik}. \tag{1.12}$$

The Ricci-identity involving the v-covariant derivative is given by [6]

$$T^i_{jk}\Big|_l\Big|_m - T^i_{jk}\Big|_m\Big|_l = T^a_{jk}S^i_{alm} - T^i_{ak}S^a_{jlm} - T^i_{ja}S^a_{klm} \tag{1.13}$$

2. P^V-Symmetric Finsler Space

Definition 2.1. A Finsler space F_n which satisfies the equation.

$$P_{ijk}\Big|_l\Big|_m - P_{ijk}\Big|_m\Big|_l = 0, \tag{2.1}$$

is defined as P^V-symmetric Finsler space. Since $P_{ijk}\Big|_l$ is completely symmetric, therefore $P_{ijk}\Big|_l\Big|_m$ is also completely symmetric for P^V-symmetric Finsler space.

Contracting (2.1) g^{ij} and using the fact that $g_{ij}\Big|_k = 0$, then (2.1) becomes

$$P_k\Big|_l\Big|_m - P_k\Big|_m\Big|_l = 0. \tag{2.2}$$

In view of the identity (1.13), the condition (2.1) reduces to :

$$P_{ajk}S^a_{ilm} + P_{iak}S^a_{jlm} + P_{ija}S^a_{klm} = 0. \tag{2.3}$$

Applying Ricci identity (1.13) to (2.2) also, we have

$$P^a_{ak}|_l|_m - P^a_{ak}|_m|_l = -P_a S^a_{klm} = 0. \tag{2.4}$$

In a two-dimensional Finsler space F_2, the v-curvature tensor S^i_{jkl} vanishes identically. Hence (2.3) holds good for a two-dimensional Finsler space F_2. For a P_2-like Finsler spaces,

$$P_{ijk} = \frac{P_i P_j P_k}{P^2} \quad if \quad P \neq 0. \tag{2.5}$$

The v-curvature tensor S^i_{jkl} of P_2-like finsler space is zero, can be seen by putting P_{ijk} and C_{ijk} from (2.5) in (1.12). Hence for a P_2-like Finsler space (2.3) satisfies. Therefore, we have

Theorem 2.1. Two *dimensional and P_2-like Finsler space are P^V-symmetric.*

3. P^V-Symmetric Properties of a P-reducible Finsler space

We shall return to consider a P-reducible Finsler space. Then from (1.8), we get

$$P_i|_j = \frac{\alpha_{|0}h_{ij}}{L} - \frac{P_i l_j}{L} - \frac{P_j l_i}{L} \tag{3.1}$$

Taking V-covariant derivative of above with respect to y^k and using (1.10a), (1.10c), (1.10d) and (3.1), we get

$$P_i\Big|_j\Big|_k = \frac{\alpha_{|0}\Big|_k h_{ij} + \alpha_{|0}h_{ij}\Big|_k}{L} - \frac{\alpha_{|0}h_{ij}L\Big|_k}{L^2}$$

$$-\left[\frac{L\left(P_i|_k l_j + P_i l_j|_k\right) - P_i l_j L|_k}{L^2}\right]$$

$$-\left[\frac{L\left(P_j|_k l_i + P_j l_i|_k\right) - P_j l_i L|_k}{L^2}\right]$$

$$P_i|_j|_k = \frac{\alpha_{|0}|_k h_{ij}}{L} - \frac{1}{L^2}\alpha_{|0}\left(h_{ij}l_k + 2h_{jk}l_i + 2h_{ki}l_j\right)$$

$$-\frac{1}{L^2}\left(P_i h_{jk} + P_j h_{ki}\right) + \frac{2}{L^2}\left(P_i l_j l_k + P_j l_k l_i + P_k l_i l_j\right) \tag{3.2}$$

From (3.2), we get

$$P_i|_j|_k - P_i|_j|_k = \left(\frac{\alpha_{|0}|_k h_{ij} - \alpha_{|0}|_j h_{ik}}{L}\right) - \frac{\alpha_{|0}}{L^2}\left(h_{ik}l_j - h_{ij}l_k\right)$$

$$-\frac{1}{L^2}\left(P_j h_{ik} - P_k h_{ij}\right) \tag{3.3}$$

For P^V-symmetric Finsler space, equation (3.3) reduces

$$L\left(\alpha_{|0}|_k h_{ij} - \alpha_{|0}|_j h_{ik}\right) - \alpha_{|0}\left(h_{ik}l_j - h_{ij}l_k\right) - \left(P_j h_{ik} - P_k h_{ij}\right) = 0$$

Contracting above result by g^{ij}, we obtain

$$L\left[(n-1)\alpha_{|0}|_k - \alpha_{|0}|_j\left(\delta^j_k - l^j l_k\right)\right] - \alpha_{|0}\left[\left(\delta^j_k - l^j l_k\right)l_j - (n-1)l_k\right]$$

$$-\left[P_j\left(\delta^j_k - l^j l_k\right) - (n-1)P_k\right] = 0.$$

Or $\quad L\left[(n-2)\alpha_{|0}|_k + \alpha_{|0}|_j l^j l_k\right] + (n-1)\alpha_{|0}l_k + (n-2)P_k = 0$

Or $\quad (n-2)L\alpha_{|0}|_k + \frac{\partial\alpha_{|0}}{\partial y^j}y^j l_k + (n-1)\alpha_{|0}l_k + (n-2)P_k = 0$

Because $\frac{\partial\alpha_{|0}}{\partial y^j}y^j = -\alpha_{|0}$, as α is a homogeneous of degree –1, therefore, we have

$$(n-2)L\alpha_{|0}|_k + (n-2)\alpha_{|0}l_k + (n-2)P_k = 0$$

Or

$$(n-2)\left[L\alpha_{|0}|_k + \alpha_{|0}l_k + P_k\right] = 0, \tag{3.4}$$

For n>2, (3.4) reduces to

$$L\alpha_{|0}|_k + \alpha_{|0}l_k + P_k = 0 \tag{3.5}$$

Proposition 3.1. If a P-reducible Finsler space is P^V-symmetric, then the scalar α, given by (1.8) satisfies (3.5).

Let us work out the condition under which a P-reducible Finsler space is also P^V-symmetric. Putting the values P_{ijk} and C_{ijk} from (1.6) and (1.7) in (1.12), then the condition (2.4) reduces to

$$P_r S^r_{ilm|0} = P_r\left[P^r_{la}C^a_{im} + C^r_{la}P^a_{im} - P^r_{ma}C^a_{il} - C^r_{ma}P^a_{il}\right] = 0$$

$$(3.6) P_r S^r_{ilm|0} = \frac{1}{(n+1)^2}P_r\left[\left(h^r_l P_a + h^r_a P_l + h_{la}P^r\right)\left(h^a_i C_m + h^a_m C_i + h_{im}C^a\right)\right.$$

$$+\left(h^r_l C_a + h^r_a C_l + h_{la}C^r\right)\left(h^a_i P_m + h^a_m P_i + h_{im}P^a\right)$$

$$-\left(h^r_m P_a + h^r_a P_m + h_{ma}P^r\right)\left(h^a_i C_l + h^a_l C_i + h_{il}C^a\right)$$

$$-\left(h^r_m C_a + h^r_a C_m + h_{ma} C^r\right)\left(h^a_i P_l + h^a_l P_i + h_{il} P^a\right)\Big] = 0$$

Or

$$P_r S^r_{ilm}|_0 = \frac{2P^2}{(n+1)^2}\left[P_l P_{im} - h_{il} P_m\right] = 0$$

Or

$$P_l h_{im} - h_{il} P_m = 0 \ \ if \ \ P^2 \neq 0 \tag{3.7}$$

Contracting (3.7) by g^{im}, we have

$$(n-2)P_l - (\delta^m_l - l^m l_l)P_m = 0$$
$$(n-2)P_l - P_l = 0$$
$$(n-2)P_l = 0 \tag{3.8}$$

for n > 2, (3.8) reduces to

$$P_l = 0 \tag{3.9}$$

It is obvious from (1.6) that if $P_i = 0$ for a P-reducible Finsler space, then $P_{ijk}=0$ immediately, so that the space is concluded to be Riemannian.

Thus, the theorem

Theorem 3.1. *If P-reducible Finsler space is P^V-symmetric, then it will be a Riemannian space. Therefore α vanishes.*

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