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Table of Contents

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On gravitational wave modes in Nash theory of gravity

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Abstract: In this Letter, we consider original Nash theory of gravity. In terms of the scalar fields representation, Nash gravity is equivalent to the action of bi-scalar tensor gravity with four derivative terms of metric tensor. We then quantify the ghost in the theory. In order to study the gravitational wave modes of the theory, we perform a small perturbation over a fixed Minkowski background. We discover the standard wave equation and obtain the solutions as those of the standard general relativity. In order to satisfy the GR in the weak field limit, we further modify the original Nash theory by adding the Einstein-Hibert term called modified Nash gravity. In this theory, we once study the gravitational wave modes. We derive equations of motion and examine the solutions. We discover the two-mode solutions: the massless graviton and the massive one. Using a uniform prior probability on the graviton mass, we can constrain mass parameter.

Keywords: 04.30, 04.30. Nk, 04.50.+h, 98.70.Vc gravitational waves; alternative theories of gravity; cosmology

1. Introduction

The recent detection of gravitational waves (GWs) from mergers of primordial black holes (BHs) or neutron stars (NSs) announced by LIGO/VIRGO [1, 2, 3, 4, 5] has pinned the new era of gravitational wave cosmology. This is a new window to probe the strong gravity physics. However, these individual two-body sources are just one of the many signatures in nature we have already detected. The GWs could be produced by other physical circumstances, for instance, during inflation [6] and reheating [7] or another exotic post-inflationary physics such as phase transitions [14, 13] or topological defects [15]. We hope that these distinct backgrounds could be detected by present and future experiments.
As mentioned in Ref. [7], a significant fraction of energy in the form of a stochastic background of gravitational waves can be produced during reheating stage. During the process, the amplitude of gravity waves exponentially grows due to the tachyonic preheating. Then a burst of gravitational radiation occurs followed by the end of gravitational waves production. In addition, cosmic defects are products of a phase transition in the early universe. The sources of GW can be of different natures, not only from quantum vacuum fluctuations in the early universe but also from astronomical sources, e.g. black hole binaries,

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supernovae, and pulsars. These different sources have their own GW characteristics. In the present work, we consider original Nash theory of gravity and examine possible gravitational wave modes of the theory.

Nash gravity and Scalar fields reduction

We start our study by considering the action of Nash gravity in four-dimensional spacetime, where the metric is a Riemanninan metric $g_{\mu\nu}$. Here the action consists of higher order corrected gravity and we adapted a notation as $g_{\mu\nu} = (-1,1,1,1), R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ and all Greek indices are spacetime indices running from 0 to 3. The original Nash action takes the form [8]

$$S_{\mathrm{N}ash} = \frac{1}{2}\int d^4x\sqrt{-g}(2R_{\mu\nu}R^{\mu\nu} - R^2). \tag{1}$$

In the above action, we have used natural units with $c = 1$ and set the gravitational coupling constant $\kappa^2 = 8\pi G/c^4 \equiv 1$. Notice that cosmology of Nash gravity was recently investigated by Refs.[9, 10, 11, 12]. Here the authors of Ref.[9] employed the Noether symmetry technique to quantify exact solutions. Recently, the matter field contents were added to the original theory [10].

We rewrite the action of theory, Eq.(1), using a pair of auxiliary fields ϕ_1, ϕ_2 in the following form

$$S = \frac{1}{2}\int d^4x\sqrt{-g}(2R_{\mu\nu}R^{\mu\nu} - R^2 + F_1(R - \phi_1) + F_2(R_{\mu\nu}R^{\mu\nu} - \phi_2)). \tag{2}$$

Here Lagrange multipliers $F_i, i = 1,2$ can be quantified by varying the total action with respect to R and $R_{\mu\nu}R^{\mu\nu}$ as variational variables. Employing this technique, we obtain

$$F_1 = \frac{\partial S}{\partial R} = 2R, \quad F_2 = \frac{\partial S}{\partial(R_{\mu\nu}R^{\mu\nu})} = -2. \tag{3}$$

Interestingly, substituting the above expressions in Eq. (??) we obtain:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}(R^2 - 2R\phi_1 + 2\phi_2)). \tag{4}$$

What we observe regarding this reduced form of the Nash action is that the matrix $\frac{\partial^2 F_i}{\partial\phi_1\partial\phi_2} = 0$ is degenerate. As a result, Nash gravity is equivalent to the action of bi-scalar-tensor gravity with four derivative terms of metric tensor. It is illustrative to present another reduced form of Nash gravity using Gauss-Bonnet invariance:

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$$\frac{\delta}{\delta g^{\mu\nu}}\int\sqrt{-g}d^4x[R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}-4R_{\mu\nu}R^{\mu\nu}+R^2]$$
$$=\frac{\delta}{\delta g^{\mu\nu}}\int\sqrt{-g}d^4xC_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}-\frac{\delta S_{Nash}}{\delta g^{\mu\nu}}, \quad (5)$$

where $C_{\alpha\beta\mu\nu}$ is the Weyl tensor. A possible alternative form for the action is given as follows:

$$S=\int\sqrt{-g}d^4x[2\phi_2-\phi_1^2+F_1(R-\phi_1)+\frac{1}{2}F_2C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}+F_2(R_{\mu\nu}R^{\mu\nu}-\phi_2))], \quad (6)$$

where again we can fix F_i by varying the total action with respect to R and $R_{\mu\nu}R^{\mu\nu}$. Then the action can be reduced as:

$$S=\int\sqrt{-g}d^4x[4\phi_2-\phi_1(\phi_1+2R)+2R^2-2R_{\mu\nu}R^{\mu\nu}]. \quad (7)$$

Looking for ghosts

In order to investigate untamed ghosts, we limit our investigations to excitations around a vacuum state where there is a potential for an exact solution to the field equations with constant Ricci scalar $R=R_0$. It is adequate to define a reduced Ricci tensor $S_{\mu\nu}$ via the following transformation:

$$S_{\mu\nu}=R_{\mu\nu}-\frac{1}{4}Rg_{\mu\nu}. \quad (8)$$

If we denote the Nash Lagrangian by $L=2R_{\mu\nu}R^{\mu\nu}-R^2$, we can expand the Nash gravity action Eq. (1) around this vacuum in Taylor series form:

$$S=\int\sqrt{-g}d^4x\left[L_0+R^2-R_0^2+\frac{1}{2}\frac{\partial^2L}{\partial S^2}|_0S^2\right]. \quad (9)$$

In the above equation, S is trace of the reduced Ricci tensor and subscript 0 means to evaluate the expression at this special vacuum solution. It is easy to show that the total action will reduce to an Einstein-Hilbert action with a cosmological constant (first and second terms in action) plus a Ricci squared term, a higher order Weyl squared term and a Gauss-Bonnet term. We can write the final form for (9) in the following compact form

$$S=\int\sqrt{-g}d^4x\left[R-2\Lambda+\frac{R^2}{6m_0^2}-\frac{C^2}{2m_2^2}\right]. \quad (10)$$

where m_0^2 is a massive spin 0 field while m_2^2 corresponds to a massive spin 2, and has a

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wrong sign of kinetic term and thus has negative energy. As a result, we need to treat it as a possible Weyl ghost field.

Unstable vacuum in higher order corrected action

Introducing an auxiliary field ϕ in action Eq. (10), we obtain:

$$S=\int \sqrt{-g}d^4x\left[\left(1+\frac{\phi}{3m_0^2}\right)R-\frac{\phi^2}{6m_0^2}-\frac{C^2}{2m_2^2}\right]. \tag{11}$$

Changing the conformal transformation from $g_{\mu\nu}$ to $\tilde{g}_{\mu\nu}=(1+\frac{\phi}{3m_0^2})g_{\mu\nu}$ and finally using another scalar field $\tilde{\phi}=\ln(1+\frac{\phi}{3m_0^2})$ we find another alternative form for action (11) in Einstein frame:

$$S=\int \sqrt{-g}d^4x[\tilde{R}-\frac{3}{2}(\tilde{\nabla}\varphi)^2-\frac{3}{2}m_0^2(1-e^{-\varphi})^2-\frac{\tilde{C}^2}{2m_2^2}]\,. \tag{12}$$

The model possesses an instability in vacuum as demonstrated in a general model of modified gravity theories [16].

Graviton propagation

In the absence of matter field Lagrangian, we can derive the equation of motion (EoM) of gravitational sector. This can be done by varying the action given in Eq.(1) with respect to the metric $g_{\mu\nu}$ in the metric formalism to obtain

$$\begin{aligned}-2RG_{\mu\nu}=\frac{1}{2}g_{\mu\nu}\left(2R_{\mu\nu}R^{\mu\nu}+R^2\right)-2(g_{\mu\nu}\,\square-\nabla_\mu\nabla_\nu)R-4R_\mu^\alpha R_{\alpha\nu}\\ -2g_{\mu\nu}\nabla_\alpha\nabla_\beta R^{\alpha\beta}-2\,\square\,R_{\mu\nu}+4\nabla_\alpha\nabla_\beta R^\alpha_{(\mu}\delta^\beta_{\nu)}\end{aligned} \tag{13}$$

Notice that we observe a fourth order EoM for metric components and second order for Ricci tensors. Here $\square=g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the usual d'Alembert operator. For simplicity, we have used symmetrized tensor representation where $T_{(\alpha\beta)}=\frac{1}{2}(T_{\alpha\beta}+T_{\beta\alpha})$. This denotes symmetrisation with respect to the pair of the indices (α,β). In this work we attempt to investigate gravitational wave modes in the Nash theory of gravity. Here we are going to derive the EoM for scalar R. Taking the trace of Eq.(13), we obtain

$$\square\,R=\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta R. \tag{14}$$

Note that Eq.(14) is a Klein-Gordon equation for the scalar field Φ:

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$$\Box\,\Phi = \frac{dV}{d\Phi}. \tag{15}$$

Here we have defined $\Phi \equiv R\ \ and\ \ dV/d\Phi \equiv \mathrm{R}HSof(14)$.

Linearization around Minkowski $\mathcal{M}_4$

In order to examine GW modes of the model, we need to perturb the above field equation over a fixed Minkowski background $\eta_{\mu\nu} = \mathrm{d}ia(-,+,+,+)$. Here we expand the metric and scalar field in the following form:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ ,$$
$$\Phi = \Phi_0 + \delta\Phi. \tag{16}$$

A first order perturbation on the Ricci scalar can be denoted by $\delta\Phi = \delta R$. Furthermore, the perturbation on the Riemann and Ricci tensors take the following form:

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu h - \Box\, h_{\mu\nu}\right),$$
$$\delta R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box\, h,$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. From Eq.(14), we obtain the Klein-Gordon equation for the scalar perturbation $\delta\Phi$ as follows:

$$\Box\,\delta\Phi = R_0\delta\Phi = m^2_{spin0}\delta\Phi\,. \tag{17}$$

Note that $R_0 = R(\eta_{\mu\nu}) = 0$. We then perturb the field equations (13) to obtain

$$\frac{1}{3}(\eta_{\mu\nu}\,\Box - \partial_\mu\partial_\nu)\delta R - 2\eta_{\mu\nu}\partial_a\partial_b\delta R^{ab} - 2\,\Box\,(\delta R_{\mu\nu}) + 2\partial_a\partial_b\delta R^a_{(\mu}\delta^b_{\nu)} = 0. \tag{18}$$

Commonly, it is more convenient to work in the Fourier space. In this case we just replace $\partial_\gamma h_{\mu\nu} \to ik_\gamma h_{\mu\nu}$ and $\Box\, h_{\mu\nu} \to -k^2 h_{\mu\nu}$. Then the above equation becomes

$$\frac{1}{3}(\eta_{\mu\nu}k^2 - k_\mu k_\nu)\delta R + 2\eta_{\mu\nu}k_a k_b\delta R^{ab} + 2k^2\delta R_{\mu\nu} - 4k_a k_b\delta R^a_{(\mu}\delta^b_{\nu)} = 0. \tag{19}$$

In the case of the metric perturbation, we write

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} + \eta_{\mu\nu}h_f, \tag{20}$$

where h_f denotes $h - \bar{h}$. Our gauge freedom takes the usual conditions $\partial_\mu\bar{h}^{\mu\nu} = 0$ and $\bar{h} = 0$. The first of these conditions implies that $k_\mu\bar{h}^{\mu\nu} = 0$ while the second dictates

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$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \eta_{\mu\nu} h_f, \tag{21}$$

$$h = 4h_f. \tag{22}$$

With these perturbations, we find [17]:

$$\delta R_{\mu\nu} = \frac{1}{2}\left(2k_\mu k_\nu h_f + k^2 \eta_{\mu\nu} h_f + k^2 \bar{h}_{\mu\nu}\right),$$
$$\delta R = 3k^2 h_f,$$
$$k_\alpha k_\beta \delta R^{\alpha\beta}_{(\mu\nu)} = -\frac{1}{2}\left((k^4 \eta_{\mu\nu} - k^2 k_\mu k_\nu) h_f + k^4 \bar{h}_{\mu\nu}\right),$$
$$k_a k_b \delta R^a_{(\mu} \delta^b_{\nu)} = \frac{3}{2} k^2 k_\mu k_\nu h_f\ . \tag{23}$$

Substituting (20)-(23) into (19) and performing some algebraic simplification, we obtain

$$k^4 \bar{h}_{\mu\nu} = (\eta_{\mu\nu} k^2 - k_\mu k_\nu)\delta R, \tag{24}$$

where we have used $R_0 = 0$. The above relation implies

$$k^4 \bar{h}_{\mu\nu} = 0\ , \tag{25}$$

where we have defined $m^2_{spin\ 2} = 0$. Note that from Eq.(25) we have a modified dispersion relation which corresponds to a massless spin-2 field ($k^2 = 0$) and a massless spin-2 ghost mode $k^2 = 0$. The solution to Eq.(25) can be parametrized by plane waves:

$$\bar{h}_{\mu\nu} = A_{\mu\nu}(\vec{p}) \cdot \exp(ik^\alpha x_\alpha) + c.c., \tag{26}$$

where

$$k^\alpha \equiv (\omega_{m_{spin2}}, \vec{p}), \quad \omega_{m_{spin2}} = p. \tag{27}$$

Here m_{spin2} is zero in the case of massless spin-2 mode and the polarization tensors is defined as $A_{\mu\nu}(\vec{p})$. Notice that we obtain the standard wave equation (25) and solutions (26) in general relativity.

Modified Nash gravity with Einstein-Hilbert term

The original Nash theory doesn't recover GR results in the weak limit. Actually, it will be very useful as a GR complementary part at UV regime and consequently very good as a potentially candidate for quantum gravity. Regarding the recent observations of GW, we are investigating a soft modification of the Nash action by adding an IR completeness term R to the action as following:

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$$S_{EH-Nash} = \frac{1}{2}\int d^4x\sqrt{-g}(\alpha R + 2R_{\mu\nu}R^{\mu\nu} - R^2). \tag{28}$$

The action is an Einstein-Hilbert corrected Nash gravity. Note that for sake or dimensionally correctness of the action we introduced a coefficient α with dimension $[\alpha] = [R] = L^{-2} = (Gev)^2$. It is very straightforward to derive EoM in the metric formalism,

$$(\alpha - 2R)G_{\mu\nu} = \frac{1}{2}g_{\mu\nu}(2R_{\mu\nu}R^{\mu\nu} + R^2) - 2(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R) - 4R_\mu^\gamma R_{\gamma\nu} - 2g_{\mu\nu}\nabla_\gamma\nabla_\theta R^{\gamma\theta} - 2\Box R_{\mu\nu} + 4\nabla_\gamma\nabla_\theta R^\gamma_{(\mu}R^\theta_{\theta)}. \tag{29}$$

Trace of the above field equation gives

$$\Box\phi = \frac{1}{9}(\alpha R - 4R_{\mu\nu}R^{\mu\nu} + R^2), \tag{30}$$

where $\phi \equiv \alpha - \frac{2}{3}R$. Now we can linearize it around flat spacetime to obtain:

$$\Box\delta\phi = \frac{\alpha}{4}\delta\phi = m_s^2\delta\phi. \tag{31}$$

For metric perturbations, we write

$$\alpha(\delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta R) = (\eta_{\mu\nu}k^2 - k_\mu k_\nu)(\delta\phi - \frac{4}{3}\delta R)2\eta_{\mu\nu}k_\gamma k_\theta\delta R^{\gamma\theta} + 2k^2\delta R_{\mu\nu} - 4k_\gamma k_\theta\delta R^\gamma_{(\mu}R^\theta_{\theta)}. \tag{32}$$

Using the same gauge freedom, we used in the previous section:

$$\partial_\mu\bar{h}^{\mu\nu} = \partial_\mu\left(h^{\mu\nu} - \frac{\bar{h}}{2}\eta^{\mu\nu} + \eta^{\mu\nu}h_f\right) = 0$$
$$\bar{h} = 0, \quad h = 4h_f, \tag{33}$$

we simplify it and find:

$$(k^2 + \frac{k^4}{m_{spin2}^2})\bar{h}_{\mu\nu} = 0, \tag{34}$$

where $m_{spin2}^2 = -\frac{\alpha}{2}$ and meanwhile we have $\Box h_f = m_s^2 h_f$. We deduce that two modes exist, one is massless graviton with dispersion relation $k^2 = 0$. the other is a massive (if $\alpha >$

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0) ghost (non -ghost if $\alpha < 0$) with dispersion relation $k^2 = -m_{spin2}^2 = \frac{\alpha}{2}$.
In case of negative α, we constrain a graviton mass using a uniform prior probability on the graviton mass [18] $m_g \subset [10^{-26}, 10^{-16}]$ to obtain

$$|\alpha| \subset [10^{-52}, 10^{-32}]. \tag{35}$$

Notice that due to the extremely small values of α the last two terms play a leading role in the modified theory.

Summary

In this work, we considered original Nash theory of gravity. In terms of the scalar fields representation, Nash gravity is equivalent to the action of bi-scalar tensor gravity with four derivative terms of metric tensor. We then quantified the ghost in the theory and found an instability in vacuum of the theory. We studied the graviton propagation by performing a small perturbation over a fixed Minkowski background. We discovered the standard wave equation and obtain the solutions as those of the standard general relativity.
In order to satisfy the GR in the weak field limit, we further modified the original Nash theory by adding the Einstein-Hibert term leading to the so-called modified Nash gravity. In this theory, we once studied the gravitational wave modes we derived equations of motion and examine the solutions. We discovered the two-mode solutions: the massless graviton and the massive one. Using a uniform prior probability on the graviton mass, we constrained mass parameter to obtain $|\alpha| \subset [10^{-52}, 10^{-32}]$.

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Some inequalities for statistical submanifolds in Kaehler-like statistical manifolds

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Abstract: The purpose of this article is to construct some geometric inequalities for statistical submanifolds in Kaehler-like statistical manifolds with constant curvature. The main aim is to establish the statistical version of B. Y. Chen inequality and bounds for generalized normalized δ-Casorati curvatures for statistical submanifolds in Kaehler-like statistical manifold and find the condition under which equality holds. Moreover, we provide some applications of the inequalities obtained.

Keywords: Kaehler-like Statistical manifold, Conjugate connection, B. Y. Chen inequality, Casorati curvatures
Mathematics Subject Classification (2010). Primary 53B05; Secondary 53B20, 53C40

1. Introduction

In 1989, the notion of statistical submanifolds was introduced and studied by Vos [13]. Though, till the date it has made very little progress due to the hardness to find classical differential geometric approaches for study of statistical submanifolds. Furuhata [7], studied statistical hypersurfaces in the space of Hessian curvature zero and provided some examples as well. In 2018, Aquib et al. [3] studied statistical submanifolds in quaternionic Kaehler-like statistical manifolds and obtained some results. Recently, some results have been published for statistical submanifolds and submersions by different geometers [4, 5, 8, 10, 12].

Furthermore, we know that Chen [6] has obtained a sharp inequality for the sectional curvature of a submanifold in a real space forms in term of the squared mean curvature and the scalar curvature. Afterward, several geometers obtained similar inequality for different submanifolds in different manifolds [9, 11] due to its rich geometric importance.

The objective of this paper is to prove B. Y. Chen inequality for statistical submanifolds in Kaehler-like statistical manifolds with constant curvature and discuss the condition under which the inequality becomes equality. Further, we obtain bounds for generalized normalized δ-Casorati curvatures and the condition for the equality case.

In fact, to have the B. Y. Chen inequality, geometers use the well-known lemma from Chen [6]

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Lemma 1.1 [6] Let $b_1, b_2, \dots, b_p, l$ be $p+1$ real numbers for $p \geq 2$ such that

$$\left(\sum_{i=1}^{p} b_i\right)^2 = (p-1)\left(\sum_{i=1}^{p} b_i^2 + l\right).$$

Then, $2b_1b_2 \geq l$ and the equality holds if and only if $b_1 + b_2 = b_3 = \cdots = b_p$.

But, in the statistical structure we are not able to use the above lemma to obtain the result. Recently, Aquib [2] overcome this problem by deriving a new Lemma in terms of second fundamental forms σ and σ^* with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ respectively. Hence, by making use of that new Lemma we obtain the desired statistical version of B. Y. Chen inequality in Kaehler-like Statistical manifold with constant curvature.

2. Statistical manifolds and their submanifolds

Let $(\overline{\mathcal{N}}, g)$ be a Riemannian manifold and $\overline{\nabla}$ and $\overline{\nabla}^*$ be torsion-free affine connections on $\overline{\mathcal{N}}$. Then the Riemannian manifold $(\overline{\mathcal{N}}, g)$ is said to be statistical if

$$Zg(X,Y) = g(\overline{\nabla}_Z X, Y) + g(X, \overline{\nabla}_Z^* Y), \tag{2.1}$$

for $X, Y, Z \in \Gamma(T\overline{\mathcal{N}})$.

Here, we note that:

1. $\overline{\nabla}$ and $\overline{\nabla}^*$ are dual connections.
2. $(\overline{\nabla}^*)^* = \overline{\nabla}$.
3. The pair $(\overline{\nabla}, g)$ is called a statistical structure.
4. Whenever $(\overline{\nabla}, g)$ is a Statistical structure on $\overline{\mathcal{N}}$, $(\overline{\nabla}^*, g)$ is also statistical structure on $\overline{\mathcal{N}}$.
5. For the dual connections $\overline{\nabla}$ and $\overline{\nabla}^*$ we always have a Levi-Civita connection $\overline{\nabla}^\circ$ on $\overline{\mathcal{N}}$ such that
$$\overline{\nabla} + \overline{\nabla}^* = 2\overline{\nabla}^\circ, \tag{2.2}$$
6. The curvature tensor fields $\overline{R}$ and $\overline{R}^*$ of $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively, satisfies
$$g(\overline{R}^*(X,Y)Z, W) = -g(Z, \overline{R}(X,Y)W). \tag{2.3}$$

Let $\overline{\mathcal{N}}$ be a $2m$-dimensional manifold and let $\mathcal{N}$ be a n-dimensional submanifolds of $\overline{\mathcal{N}}$. Then, the corresponding Gauss formulae are [13]:

$$\begin{cases} \overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \\ \overline{\nabla}_X^* Y = \nabla_X^* Y + \sigma^*(X,Y), \end{cases}$$

where σ and σ^* are symmetric and bilinear, called imbedding curvature tensor of $\mathcal{N}$ in

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$\overline{\mathcal{N}}$ for $\overline{\nabla}$ and the imbedding curvature tensor of $\mathcal{N}$ in $\overline{\mathcal{N}}$ for $\overline{\nabla}^*$, respectively. Let us denote the normal bundle of $\mathcal{N}$ by $\Gamma(T\mathcal{N}^{\perp})$. Since σ and σ^* are bilinear, we have the linear transformations A_ξ and A^*_ξ defined by

$$\begin{cases} g(A_\xi X, Y) = g(\sigma(X,Y), \xi), \\ g(A^*_\xi X, Y) = g(\sigma^*(X,Y), \xi), \end{cases}$$

for any $\xi \in \Gamma(T\mathcal{N}^{\perp})$ and $X, Y \in \Gamma(T\mathcal{N})$. The corresponding Wiengarten formulas are [13]:

$$\begin{cases} \overline{\nabla}_X \xi = -A^*_\xi X + \nabla^{\perp}_X \xi, \\ \overline{\nabla}^*_X \xi = -A_\xi X + \nabla^{*\perp}_X \xi, \end{cases}$$

for any $\xi \in \Gamma(T\mathcal{N}^{\perp})$ and $X \in \Gamma(T\mathcal{N})$. The connections $\nabla^{\perp}_X$ and $\nabla^{*\perp}_X$ given in the above equations are Riemannian dual connections with respect to the induced metric on $\Gamma(T\mathcal{N}^{\perp})$. The corresponding Gauss equations are defined as [13]

$$\begin{aligned} g(\overline{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma^*(Y,W)) \\ -g(\sigma^*(X,W),\sigma(Y,Z)), \end{aligned} \qquad (2.4)$$

and

$$\begin{aligned} g(\overline{R}^*(X,Y)Z,W) = g(R^*(X,Y)Z,W) + g(\sigma^*(X,Z),\sigma(Y,W)) \\ -g(\sigma(X,W),\sigma^*(Y,Z)). \end{aligned} \qquad (2.5)$$

Definition 2.1 ([10]). A $2m$-dimensional statistical manifold $\overline{\mathcal{N}}$ is said to be a Kaehler-like statistical manifold with constant curvature if it admits an endomorphism J over the tangent bundle $\Gamma(\overline{\mathcal{N}})$ and a metric g and a fundamental form ω given by $\omega(X,Y) = g(X, JY)$ such that

$$J^2 = -Id; \quad \overline{\nabla}\omega = 0, \qquad (2.6)$$

for any vector fields $X, Y \in \Gamma(\overline{\mathcal{N}})$. Since ω is skew-symmetric, we have

$$g(X, JY) = -g(J^* X, Y).$$

Definition 2.2 ([10]). A Kaehler-like statistical manifold with constant curvature $\overline{\mathcal{N}}$ is said to be of constant holomorphic curvature $c \in R$ if the following curvature equation holds:

$$\begin{aligned} \overline{R}(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(X,JZ)JY \\ -g(Y,JZ)JX + g(X,JY)JZ - g(JX,Y)JZ\}. \end{aligned} \qquad (2.7)$$

Definition 2.3 ([8]). A CR-submanifold $\mathcal{N}$ of $\overline{\mathcal{N}}$ is called a totally real submanifold if $dim D_x = 0, x \in \mathcal{N}$.

Let $\mathcal{N}$ be a Riemannian manifold and $K(\pi)$ denotes the sectional curvature of $\mathcal{N}$ of the plane section $\pi \subset T_p\mathcal{N}$ at a point $p \in \mathcal{N}$. If $\{e_1, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{2m}\}$ be the orthonormal basis of $T_p\mathcal{N}$ and $T^{\perp}_p\mathcal{N}$ at any $p \in \mathcal{N}$, then the scalar curvature τ at that

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point is given by

$$\tau(p) = \sum_{1\le i<j\le n} K(e_i \wedge e_j) \tag{2.8}$$

Let $\{e_1, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{2m}\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on $\mathcal{N}$. The mean curvature vector fields are given by

$$\mathcal{H} = \frac{1}{n}\sum_{i=1}^{n} \sigma(e_i, e_i), \tag{2.9}$$

$$\mathcal{H}^* = \frac{1}{n}\sum_{i=1}^{n} \sigma^*(e_i, e_i), \tag{2.10}$$

and

$$\mathcal{H}^\circ = \frac{1}{n}\sum_{i=1}^{n} \sigma^\circ(e_i, e_i). \tag{2.11}$$

We also set

$$\| \sigma \|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)), \tag{2.12}$$

$$\| \sigma^* \|^2 = \sum_{i,j=1}^{n} g(\sigma^*(e_i, e_j), \sigma^*(e_i, e_j)), \tag{2.13}$$

and

$$\| \sigma^\circ \|^2 = \sum_{i,j=1}^{n} g(\sigma^\circ(e_i, e_j), \sigma^\circ(e_i, e_j)). \tag{2.14}$$

Further, we denote by

$$\| P \|^2 = \sum_{i,j=1}^{n} g^2(Je_j, e_i). \tag{2.15}$$

Let $K(\pi)$ denote the sectional curvature of a Riemannian manifold $\mathcal{N}$ of the plane section $\pi \subset T_p\mathcal{N}$ at a point $p \in \mathcal{N}$. Then,

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$$\tau(p) = \sum_{1\le i<j\le n} K(e_i \wedge e_j), \tag{2.16}$$

where τ is the scalar curvature. The normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{n(n-1)}. \tag{2.17}$$

We also put

$$\sigma_{ij}^{\gamma} = g(\sigma(e_i, e_j), e_\gamma), \quad \sigma_{ij}^{*\gamma} = g(\sigma^*(e_i, e_j), e_\gamma),$$

$i, j \in 1,2,..,n, \gamma \in \{n+1, \ldots, 2m\}$. The squared norms of the second fundamental form σ and σ^* are denoted by $\mathcal{C}$ and $\mathcal{C}^*$, respectively, and are given as

$$\mathcal{C} = \frac{1}{n}\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{n}(\sigma_{ij}^{\gamma})^2 \quad \text{and} \quad \mathcal{C}^* = \frac{1}{n}\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{n}(\sigma_{ij}^{*\gamma})^2, \tag{2.18}$$

called Casorati curvatures of the submanifold [14, 16].

Let L_r be an r-dimensional subspace of $T\mathcal{N}$, $r \geq 2$, and $\{e_1, e_2, \ldots, e_r\}$ is an orthonormal basis of L_r. Then

$$\tau(L_r) = \sum_{1\le \gamma<\beta\le r} K(e_\gamma \wedge e_\beta),$$

is called the scalar curvature of the r-plane section. The Casorati curvatures $\mathcal{C}$ and $\mathcal{C}^*$ of that r-plane section are [14, 16]:

$$\mathcal{C}(L_r) = \frac{1}{r}\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{r}(\sigma_{ij}^{\gamma})^2 \quad \text{and} \quad \mathcal{C}^*(L_r) = \frac{1}{r}\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{r}(\sigma_{ij}^{*\gamma})^2. \tag{2.19}$$

The normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ are defined as [15, 16]:

$$[\delta_c(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n} inf\{\mathcal{C}(L_{n-1}) | L_{n-1}: \text{ahyperplaneof} T_p\mathcal{N}\} \tag{2.20}$$

and

$$[\hat{\delta}_c(n-1)]_p = 2\mathcal{C}_p - \frac{2n-1}{2n} sup\{\mathcal{C}(L_{n-1}) | L_{n-1}: \text{a hyperplane of } T_p\mathcal{N}\}. \tag{2.21}$$

Similarly, the dual normalized δ^*-Casorati curvatures $\delta_c^*(n-1)$ and $\hat{\delta}_c^*(n-1)$ are defined as [3, 14]:

$$[\delta_c^*(n-1)]_p = \frac{1}{2}\mathcal{C}_p^* + \frac{n+1}{2n} inf\{\mathcal{C}^*(L_{n-1}) | L_{n-1}: \text{a hyperplane of } T_p\mathcal{N}\} \tag{2.22}$$

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and

$$[\hat{\delta}_c^*(n-1)]_p = 2\mathcal{C}^*{}_p - \frac{2n-1}{2n} sup\{\mathcal{C}^*(L_{n-1})|L_{n-1}\text{: a hyperplane of } T_p\mathcal{N}\}. \quad (2.23)$$

For a positive real number $t \neq n(n-1)$, put

$$b(t) = \frac{1}{nt}(n-1)(n+t)(n^2-n-t), \quad (2.24)$$

then the generalized normalized δ-Casorati curvatures $\delta_c(t;n-1)$ and $\hat{\delta}_c(t;n-1)$ are given as [15, 16]:

$$[\delta_c(t;n-1)]_p = t\mathcal{C}_p + b(t) inf\{\mathcal{C}(L_{n-1})|L_{n-1}\text{: a hyperplane of } T_p\mathcal{N}\}$$

if $0 < t < n(n+1)$, and

$$[\hat{\delta}_c(t;n-1)]_p = t\mathcal{C}_p + b(t) sup\{\mathcal{C}(L_{n-1})|L_{n-1}\text{: a hyperplane of } T_p\mathcal{N}\}$$

if $t > n(n-1)$.

Further, the dual generalized normalized δ^*-Casorati curvatures $\delta_c^*(t;n-1)$ and $\hat{\delta}_c^*(t;n-1)$ are given as [3, 14]:

$$[\delta_c^*(t;n-1)]_p = t\mathcal{C}_p^* + b(t) inf\{\mathcal{C}^*(L_{n-1})|L_{n-1}\text{: a hyperplane of } T_p\mathcal{N}\}$$

if $0 < t < n(n-1)$, and

$$[\hat{\delta}_c^*(t;n-1)]_p = t\mathcal{C}_p^* + b(t) sup\{\mathcal{C}^*(L_{n-1})|L_{n-1}\text{: ahyperplaneof} T_p\mathcal{N}\}$$

if $t > n(n-1)$.

Lemma 2.4 ([16]) Let

$$F = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = k\}$$

be a hyperplane of $\mathbb{R}^n$, and $f: \mathbb{R}^n \to \mathbb{R}$ a quadratic form given by

$$f(x_1, \dots, x_n) = a \sum_{i=1}^{n-1} (x_i)^2 + b(x_n)^2 - 2 \sum_{1 \le i < j \le n} x_i x_j, \quad a > 0, b > 0.$$

Then f has a global solution,

$$\begin{cases} x_1 = x_2 = \cdots = x_{n-1} = \frac{1}{a+1} k, \\ x_n = \frac{1}{b+1} k, \end{cases} \quad (2.25)$$

provided

$$b = \frac{n-1}{a-n+2}.$$

3. A general inequality and applications

Let $\overline{\mathcal{N}}(c)$ be a $2m$-dimensional Kaehler-like statistical manifold of constant curvature $c \in$

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R and $\mathcal{N}$ be an n-dimensional statistical submanifold of $\overline{\mathcal{N}}(c)$.
Then we obtain the following inequality and some of its applications.

Theorem 3.1 Let $\mathcal{N}$ be a statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. Then

$$2\tau \geq \frac{c}{4}\{n(n-1)+(trJ)^2+3\parallel P\parallel^2\}-\parallel\sigma\parallel\parallel\sigma^*\parallel+n^2g(\mathcal{H}^*,\mathcal{H}). \tag{3.1}$$

Proof. Let $\{e_1,\dots,e_n\}$ and $\{e_{n+1},\dots,e_{2m}\}$ be the orthonormal basis of tangent orthonormal frame and normal orthonormal frame, respectively, on $\mathcal{N}$. Putting $X=W=e_i$ and $Y=Z=e_j$ in equation (2.7) and take $i\neq j$, we have

$$\begin{aligned}\overline{R}(e_i,e_j,e_i,e_j)=\frac{c}{4}\{&g(e_j,e_j)g(e_i,e_i)-g(e_i,e_j)g(e_j,e_i)\\&+g(Je_j,e_j)g(Je_i,e_i)-g(Je_i,e_j)g(Je_j,e_i)\\&+g(e_i,Je_j)g(Je_j,e_i)-g(Je_i,e_j)g(Je_j,e_i)\}.\end{aligned} \tag{3.2}$$

On the other hand, from Gauss equation

$$\begin{aligned}R(e_i,e_j,e_i,e_j)=\overline{R}(e_i,e_j,e_i,e_j)&-g(\sigma(e_i,e_j),\sigma^*(e_j,e_i))\\&+g(\sigma^*(e_i,e_i),\sigma(e_j,e_j)).\end{aligned} \tag{3.3}$$

Now, combining (3.2) and (3.3) and taking the summation over $1\leq i,j\leq n$, we find

$$\begin{aligned}\sum_{i,j=1}^{n}R(e_i,e_j,e_j,e_i)=\frac{c}{4}\{n^2-(trJ)^2+2\parallel P\parallel^2\}&-\sum_{i,j=1}^{n}g(\sigma(e_i,e_j),\sigma^*(e_j,e_i))\\&+\sum_{i,j=1}^{n}g(\sigma^*(e_i,e_i),\sigma(e_j,e_j)).\end{aligned} \tag{3.4}$$

Equation (3.4) implies

$$2\tau=\frac{c}{4}\{n^2-(trJ)^2+2\parallel P\parallel^2\}-\sum_{\alpha=n+1}^{2m}\sum_{i,j=1}^{n}\sigma_{ij}^{\alpha}\sigma_{ji}^{\alpha*}+n^2g(\mathcal{H}^*,\mathcal{H}) \tag{3.5}$$

$$\geq\frac{c}{4}\{n^2-(trJ)^2+2\parallel P\parallel^2\}-\parallel\sigma\parallel\parallel\sigma^*\parallel+n^2g(\mathcal{H}^*,\mathcal{H}). \tag{3.6}$$

An immediate consequence of the Theorem 3.1 is the following results.

Corollary 3.2. Let $\mathcal{N}$ be a statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. If θ be the angle between $\mathcal{H}$ and $\mathcal{H}^*$, then

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Angle between $\mathcal{H}$ and $\mathcal{H}^*$	Inequalities
θ	$2\tau \geq \frac{c}{4}\{n(n-1)+(trJ)^2+3\parallel P\parallel^2\}-\parallel\sigma\parallel\parallel\sigma^*\parallel$ $+n^2\|\|\mathcal{H}^*\|\|\|\|\mathcal{H}\|\|\cos\theta$
$0°$(i.e. $\mathcal{H}\ \|\|\ \mathcal{H}^*$)	$2\tau \geq \frac{c}{4}\{n(n-1)+(trJ)^2+3\parallel P\parallel^2\}-\parallel\sigma\parallel\parallel\sigma^*\parallel$ $+n^2\|\|\mathcal{H}^*\|\|\|\|\mathcal{H}\|\|$
$90°$(i.e. $\mathcal{H}\perp\mathcal{H}^*$)	$2\tau \geq \frac{c}{4}\{n(n-1)+(trJ)^2+3\parallel P\parallel^2\}-\parallel\sigma\parallel\parallel\sigma^*\parallel$

Corollary 3.3 Let $\mathcal{N}$ be a statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. Then

$$\tau \geq \frac{c}{8}\{n^2-(trJ)^2+2\parallel P\parallel^2\}. \quad (3.7)$$

If

1. either $\mathcal{N}$ is totally geodesic for the connection $\overline{\nabla}$ or
2. $\mathcal{N}$ is totally geodesic for the connection $\overline{\nabla}^*$.

Proof. If $\mathcal{N}$ totally geodesic for the connection ∇, then $\parallel\sigma\parallel=0$, this implies $\mathcal{H}=0$. If $\mathcal{N}$ totally geodesic for the connection ∇^*, then $\parallel\sigma^*\parallel=0$, this implies $\mathcal{H}^*=0$. Hence, we have our assertions with the help of (3.1).

We also notice the following.

Corollary 3.4 Let $\mathcal{N}$ be a statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. Then

$$2\tau \geq \frac{c}{4}n^2-\parallel\sigma\parallel\parallel\sigma^*\parallel+n^2g(\mathcal{H}^*,\mathcal{H}), \quad (3.8)$$

if $\mathcal{N}$ is totally real.
Proof. We obtain the result using definition 2.3 in equation (3.6).

We also notice the following result.

Theorem 3.5 Let $\mathcal{N}$ be a totally real statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. Then

$$\tau \geq \frac{c}{8}n^2. \quad (3.9)$$

If

1. either $\mathcal{N}$ is totally geodesic for the connection $\overline{\nabla}$, or

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2. $\mathcal{N}$ is totally geodesic for the connection $\overline{\nabla}^*$.

Proof. If $\mathcal{N}$ totally geodesic for the connection ∇, then $\| \sigma \| = 0$, this implies $\mathcal{H} = 0$. If $\mathcal{N}$ totally geodesic for the connection ∇^*, then $\| \sigma^* \| = 0$, this implies $\mathcal{H}^* = 0$. Hence, we have our assertions with the help of (3.8).

4. B. Y. Chen inequality

Let $\overline{\mathcal{N}}(c)$ be a $2m$-dimensional Kaehler-like statistical manifold of constant curvature $c \in R$ and $\mathcal{N}$ be an n-dimensional statistical submanifold of $\overline{\mathcal{N}}(c)$.

Then we have the following.

Theorem 4.1 Let $\mathcal{N}$ be a statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$ with

$$\sum_{\alpha=n+1}^{2m} \sigma_{11}^{*\alpha}\sigma_{22}^{\alpha} = \sum_{\alpha=n+1}^{2m} \sigma_{12}^{\alpha}\sigma_{12}^{*\alpha}.$$

Then

$$K(\Pi) \leq 2\tau + \frac{c}{4}\{2\Theta(\pi) + (trJ)^2 - n^2 - g(Je_2, e_2)g(Je_1, e_1) - 2 \| P \|^2\}$$
$$+\frac{1}{2}n^2(\| \mathcal{H} \|^2 + \| \mathcal{H}^* \|^2) - 2n^2 \| H^\circ \|^2 + \| \sigma \| \| \sigma^* \|, \tag{4.1}$$

where $\Theta(\pi) = g^2(Je_2, e_1)$. Moreover, the equality holds if

$$\sigma = k\sigma^*, \quad k \in \mathbb{R}^+$$

Proof. Equation (3.5) can be rewritten as

$$2\tau = \frac{c}{4}\{n^2 - (trJ)^2 + 2 \| P \|^2\}$$
$$-\sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma^*(e_j, e_i)) + n^2 g(\mathcal{H}^*, \mathcal{H}) \tag{4.2}$$

$$= \frac{c}{4}\{n^2 - (trJ)^2 + 2 \| P \|^2\} - \frac{1}{2}\{g(\sigma(e_i, e_j) + \sigma^*(e_j, e_i), \sigma(e_i, e_j)$$
$$+\sigma^*(e_j, e_i)) - g(\sigma(e_i, e_j), \sigma(e_i, e_j)) - g(\sigma^*(e_j, e_i), \sigma^*(e_j, e_i))\}$$
$$+\frac{n^2}{2}\{g(\mathcal{H}^* + \mathcal{H}, \mathcal{H}^* + \mathcal{H}) - g(\mathcal{H}, \mathcal{H}) - g(\mathcal{H}^*, \mathcal{H}^*)\}. \tag{4.3}$$

Indeed, from equation (2.2), $2\mathcal{H}^\circ = \mathcal{H} + \mathcal{H}^*$. Then it follows from the above equation that

$$2\tau = \frac{c}{4}\{n^2 - (trJ)^2 + 2 \| P \|^2\} - 2g(\sigma^\circ(e_i, e_j), \sigma^\circ(e_i, e_j)) \tag{4.4}$$

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$$+\frac{1}{2}(\| \sigma \|^2 + \| \sigma^* \|^2) + 2n^2 g(\mathcal{H}^\circ, \mathcal{H}^\circ) - \frac{n^2}{2} g(\mathcal{H}, \mathcal{H}) - \frac{n^2}{2} g(\mathcal{H}^*, \mathcal{H}^*).$$

Also, we know that $K(\Pi) = R(e_1, e_2, e_1, e_2)$.

Therefore,

$$\begin{aligned}
K(\Pi) &= \overline{R}(e_1, e_2, e_1, e_2) - g(\sigma(e_1, e_2), \sigma^*(e_2, e_1)) + g(\sigma^*(e_1, e_1), \sigma(e_2, e_2)) \\
&= \frac{c}{4}\{1 - g(Je_2, e_2) g(Je_1, e_1) + 2g^2(e_1, Je_2) - g(Je_1, e_2) g(Je_2, e_1)\} \\
&\quad + \sum_{\alpha=n+1}^{2m} [\sigma_{11}^{*\alpha}\sigma_{22}^{\alpha} - \sigma_{12}^{\alpha}\sigma_{12}^{*\alpha}] \\
&= \frac{c}{4}\{-g(Je_2, e_2) g(Je_1, e_1) + 2g^2(e_1, Je_2)\} + \sum_{\alpha=n+1}^{2m} [\sigma_{11}^{*\alpha}\sigma_{22}^{\alpha} - \sigma_{12}^{\alpha}\sigma_{12}^{*\alpha}] \\
&= \frac{c}{4}\{-g(Je_2, e_2) g(Je_1, e_1) + 2\Theta(\pi)\} + \sum_{\alpha=n+1}^{2m} [\sigma_{11}^{*\alpha}\sigma_{22}^{\alpha} - \sigma_{12}^{\alpha}\sigma_{12}^{*\alpha}]. \qquad (4.5)
\end{aligned}$$

Taking into account (3.4) and (4.5), we get

$$\begin{aligned}
K(\Pi) - 2\tau = &\frac{c}{4}\{-g(Je_2, e_2) g(Je_1, e_1) + 2\Theta(\pi) - n^2 + (trJ)^2 - 2\| P \|^2\} \\
&+ \sum_{\alpha=n+1}^{2m} [\sigma_{11}^{*\alpha}\sigma_{22}^{\alpha} - \sigma_{12}^{\alpha}\sigma_{12}^{*\alpha}] - 2n^2 g(\mathcal{H}^\circ, \mathcal{H}^\circ) + \frac{n^2}{2} g(\mathcal{H}, \mathcal{H}) \\
&+ \frac{n^2}{2} g(\mathcal{H}^*, \mathcal{H}^*) + 2\| \sigma^\circ \|^2 - \frac{1}{2}(\| \sigma \|^2 + \| \sigma^* \|^2) \qquad (4.6)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\| \sigma + \sigma^* \|^2 &= g(\sigma + \sigma^*, \sigma + \sigma^*) \\
&= \| \sigma \|^2 + g(\sigma, \sigma^*) + g(\sigma^*, \sigma) + \| \sigma^* \|^2 \\
&= \| \sigma \|^2 + 2g(\sigma, \sigma^*) + \| \sigma^* \|^2 \\
&\leq \| \sigma \|^2 + 2\| \sigma \| \| \sigma^* \| + \| \sigma^* \|^2, \qquad (4.7)
\end{aligned}$$

and the equality in the above inequality holds if

$$\sigma = k\sigma^*, \quad k \in \mathbb{R}^+. \qquad (4.8)$$

Equation (4.7) implies

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$$\| \sigma \|^2 + \| \sigma^* \|^2 \geq \| \sigma + \sigma^* \|^2 - 2 \| \sigma \| \| \sigma^* \|. \tag{4.9}$$

Using equation (4.9) in (4.6), we find

$$\begin{aligned} K(\Pi) - 2\tau \leq & \frac{c}{4}\{-g(Je_2, e_2)g(Je_1, e_1) + 2\Theta(\pi) - n^2 + (trJ)^2 - 2 \| P \|^2\} \\ & + \sum_{\alpha=n+1}^{2m} [\sigma_{11}^{*\alpha}\sigma_{22}^{\alpha} - \sigma_{12}^{\alpha}\sigma_{12}^{*\alpha}] - 2n^2 g(\mathcal{H}^\circ, \mathcal{H}^\circ) + \frac{n^2}{2} g(\mathcal{H}, \mathcal{H}) \\ & + \frac{n^2}{2} g(\mathcal{H}^*, \mathcal{H}^*) + 2 \| \sigma^\circ \|^2 - \frac{1}{2} \| \sigma^2 + \sigma^* \|^2 + \| \sigma \| \| \sigma^* \| \\ & = \frac{c}{4}\{-g(Je_2, e_2)g(Je_1, e_1) + 2\Theta(\pi) - n^2 + (trJ)^2 - 2 \| P \|^2\} \\ & + \sum_{\alpha=n+1}^{2m} [\sigma_{11}^{*\alpha}\sigma_{22}^{\alpha} - \sigma_{12}^{\alpha}\sigma_{12}^{*\alpha}] - 2n^2 g(\mathcal{H}^\circ, \mathcal{H}^\circ) + \frac{n^2}{2} g(\mathcal{H}, \mathcal{H}) \\ & + \frac{n^2}{2} g(\mathcal{H}^*, \mathcal{H}^*) + \| \sigma \| \| \sigma^* \|. \end{aligned} \tag{4.10}$$

Using hypothesis of the theorem in (4.10), we have

$$\begin{aligned} K(\Pi) \leq & 2\tau + \frac{c}{4}\{-g(Je_2, e_2)g(Je_1, e_1) + 2\Theta(\pi) - n^2 + (trJ)^2 - 2 \| P \|^2\} \\ & -2n^2 g(\mathcal{H}^\circ, \mathcal{H}^\circ) + \frac{n^2}{2} g(\mathcal{H}, \mathcal{H}) + \frac{n^2}{2} g(\mathcal{H}^*, \mathcal{H}^*) + \| \sigma \| \| \sigma^* \|. \end{aligned} \tag{4.11}$$

Hence, we have our assertion.
The following results follows from Theorem 4.1.

Corollary 4.2 Let $\mathcal{N}$ be a statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. If θ be the angle between $\mathcal{H}$ and $\mathcal{H}^*$, then

Angle between $\mathcal{H}$ and $\mathcal{H}^*$	Inequalities
θ	$K(\Pi) \leq 2\tau + \frac{c}{4}\{2\Theta(\pi) + (trJ)^2 - n^2 - g(Je_2, e_2)g(Je_1, e_1) - 2 \| P \|^2\} - n^2 \|\mathcal{H}\| \|\mathcal{H}^*\| \cos\theta + \| \sigma \| \| \sigma^* \|$
0°(i.e. $\mathcal{H} \parallel \mathcal{H}^*$)	$K(\Pi) \leq 2\tau + \frac{c}{4}\{2\Theta(\pi) + (trJ)^2 - n^2 - g(Je_2, e_2)g(Je_1, e_1) - 2 \| P \|^2\} - n^2 \|\mathcal{H}\| \|\mathcal{H}^*\| + \| \sigma \| \| \sigma^* \|$
90°(i.e. $\mathcal{H} \perp \mathcal{H}^*$)	$K(\Pi) \leq 2\tau + \frac{c}{4}\{2\Theta(\pi) + (trJ)^2 - n^2 - g(Je_2, e_2)g(Je_1, e_1) - 2 \| P \|^2\} + \| \sigma \| \| \sigma^* \|$

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Corollary 4.3 Let $\mathcal{N}$ be a statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. Then

$$K(\Pi) \leq 2\tau + \frac{c}{4}\{2\Theta\pi + (trJ)^2 - n^2 - g(Je_2, e_2)g(Je_1, e_1) - 2 \parallel P \parallel^2\}. \quad (4.12)$$

If

1. either $\mathcal{N}$ is totally geodesic for the connection $\overline{\nabla}$ or
2. $\mathcal{N}$ is totally geodesic for the connection $\overline{\nabla}^*$.

Proof. If $\mathcal{N}$ totally geodesic for the connection $\overline{\nabla}$, then $\sigma = 0$, this implies $\mathcal{H} = 0$. If $\mathcal{N}$ totally geodesic for the connection $\overline{\nabla}^*$, then $\sigma^* = 0$, this implies $\mathcal{H}^* = 0$. We also have $2\mathcal{H}^\circ = \mathcal{H} + \mathcal{H}^*$. Taking these facts into consideration and combining with (4.1) we have the required result.

Theorem 4.4 Let $\mathcal{N}$ be a totally real statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. Then

$$K(\Pi) \leq 2\tau + \frac{c}{4}\{(1 + n - n^2)\} + \frac{1}{2}n^2(\parallel \mathcal{H} \parallel^2 + \parallel \mathcal{H}^* \parallel^2) - 2 \parallel \mathcal{H}^\circ \parallel^2 + \parallel \sigma \parallel \parallel \sigma^* \parallel. \quad (4.13)$$

Proof. We proof the result using same process as in case of Theorem 4.1 and the definition 2.3.

Corollary 4.5 Let $\mathcal{N}$ be a totally real statistical submanifold in Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. Then

$$K(\Pi) \leq 2\tau + \frac{c}{4}\{(1 + n - n^2)\}. \quad (4.14)$$

If

1. either $\mathcal{N}$ is totally geodesic for the connection $\overline{\nabla}$ or
2. $\mathcal{N}$ is totally geodesic for the connection $\overline{\nabla}^*$.

Proof. If $\mathcal{N}$ totally geodesic for the connection $\overline{\nabla}$, then $\sigma = 0$, this implies $\mathcal{H} = 0$. If $\mathcal{N}$ totally geodesic for the connection $\overline{\nabla}^*$, then $\sigma^* = 0$, this implies $\mathcal{H}^* = 0$. We also have $2\mathcal{H}^\circ = \mathcal{H} + \mathcal{H}^*$. Taking these facts into consideration and combining with (4.13) we have the required result.

Remark 4.6 The similar results can be obtain for $\overline{R}^*$ as well.

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5. Generalized Normalized δ-Casorati Curvature

In this section, we mainly show that the normalized scalar curvature is bounded above by the generalized normalized δ-Casorati curvatures for statistical submanifolds of Kaehler-like statistical manifold with constant curvature c. We mainly proof the following result.

Theorem 5.1 Let $\mathcal{N}$ be a statistical submanifold in a Kaehler-like statistical manifold $\overline{\mathcal{N}}(c)$. Then, the generalized normalized δ-Casorati curvatures $\delta_c(t;n-1)$ and $\delta_c^*(t;n-1)$ satisfy

$$\rho \leq \frac{2}{n(n-1)}\delta_c^\circ(t;n-1)+\frac{c}{4n(n-1)}\{n^2-(trJ)^2+2\parallel P\parallel^2\} \\ +\frac{1}{(n-1)}\mathcal{C}^\circ-\frac{n}{2(n-1)}(\parallel \mathcal{H}\parallel^2+\parallel \mathcal{H}^*\parallel^2), \tag{5.1}$$

for real t, $0<t<n(n-1)$, where $2\delta_c^\circ(t;n-1)=\delta_c(t;n-1)+\delta_c^*(t;n-1)$ and $2\mathcal{C}^\circ=\mathcal{C}+\mathcal{C}^*$. The equality case holds in Equation (5.1) if and only if the component of σ satisfies

$$\begin{cases}\sigma_{ij}^{\circ\gamma}=0, \quad i,j\in\{1,\dots,n\}, \quad \alpha\in\{n+1,\dots,2m\},\\ \sigma_{11}^{\circ\gamma}=\sigma_{22}^{\circ\gamma}=\cdots=\sigma_{n-1n-1}^{\circ\gamma}=\dfrac{t}{n(n-1)}\sigma_{nn}^{\circ\gamma}, \quad \alpha\in\{n+1,\dots,2m\}.\end{cases} \tag{5.2}$$

Proof. Equation (4.4) can be re-written as

$$2\tau=\frac{c}{4}\{n^2-(trJ)^2+2\parallel P\parallel^2\}-2n\mathcal{C}^\circ \\ +\frac{n}{2}(\mathcal{C}+\mathcal{C}^*)+2n^2g(\mathcal{H}^\circ,\mathcal{H}^\circ)-\frac{n^2}{2}g(\mathcal{H},\mathcal{H})-\frac{n^2}{2}g(\mathcal{H}^*,\mathcal{H}^*). \tag{5.3}$$

Now, we may assume a quadratic polynomial $\mathcal{T}$ as

$$\mathcal{T}=2tC^\circ+2b(t)C^\circ(L_{n-1})+\frac{c}{4}\{n^2-(trJ)^2+2\parallel P\parallel^2\} \\ -2\tau+\frac{n}{2}(\mathcal{C}+\mathcal{C}^*)-\frac{n^2}{2}(\parallel \mathcal{H}\parallel^2+\parallel \mathcal{H}^*\parallel^2), \tag{5.4}$$

where L_{n-1} is the hyperplane of T_pM. Without loss of generality, let us assume that L_{n-1} is spanned by $e_1,\dots,e_{n-1}$, then from Equation (5.4) it follows that

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$$\mathcal{T} = 2\left(\frac{t+n}{n}\right)\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{n}(\sigma_{ij}^{\circ\gamma})^2 + 2\frac{b(t)}{n}\sum_{\gamma=n+1}^{2m}\sum_{i=1}^{n-1}(\sigma_{ii}^{\circ\gamma})^2$$
$$-2\sum_{\gamma=n+1}^{2m}\sum_{i<j=1}^{n}\sigma_{ii}^{\circ\gamma}\sigma_{jj}^{\circ\gamma}$$

which implies

$$\frac{\mathcal{T}}{2} = \left(\frac{t+n}{n}\right)\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{n}(\sigma_{ij}^{\circ\gamma})^2 + \frac{b(t)}{n}\sum_{\gamma=n+1}^{2m}\sum_{i=1}^{n-1}(\sigma_{ii}^{\circ\gamma})^2 - \sum_{\gamma=n+1}^{2m}\sum_{i<j=1}^{n}\sigma_{ii}^{\circ\gamma}\sigma_{jj}^{\circ\gamma},$$
$$= \left(\frac{t}{n}+\frac{b(t)}{n-1}\right)\sum_{\gamma=n+1}^{2m}\sum_{i=1}^{n-1}(\sigma_{ii}^{\circ\gamma})^2$$
$$+2\left(\frac{t}{n}+\frac{b(t)}{n-1}+1\right)\sum_{\gamma=n+1}^{2m}\sum_{i<j=1}^{n-1}(\sigma_{ij}^{\circ\gamma})^2 + \left(\frac{t}{n}\right)\sum_{\gamma=n+1}^{2m}(\sigma_{nn}^{\circ\gamma})^2$$
$$+2\left(\frac{t}{n}+1\right)\sum_{\gamma=n+1}^{2m}\sum_{i=1}^{n-1}(\sigma_{in}^{\circ\gamma})^2 - 2\sum_{\gamma=n+1}^{2m}\sum_{i<j=1}^{n}\sigma_{ii}^{\circ\gamma}\sigma_{jj}^{\circ\gamma},$$
$$\geq \sum_{\gamma=n+1}^{2m}[\left(\frac{t}{n}+\frac{b(t)}{n-1}\right)\sum_{i=1}^{n-1}(\sigma_{ii}^{\circ\gamma})^2 + \frac{t}{n}(\sigma_{nn}^{\circ\gamma})^2 - 2\sum_{i<j=1}^{n}\sigma_{ii}^{\circ\gamma}\sigma_{jj}^{\circ\gamma}].$$

Now, we consider the quadratic forms $f_\gamma: \mathbb{R}^n \to \mathbb{R}$ such that

$$f_\gamma(\sigma_{11}^{\circ\gamma}, \sigma_{22}^{\circ\gamma}, \dots, \sigma_{n+1n+1}^{\circ\gamma})$$
$$= \sum_{\gamma=n+1}^{2m}\left[\left(\frac{t}{n}+\frac{b(t)}{n-1}\right)\sum_{i=1}^{n-1}(\sigma_{ii}^{\circ\gamma})^2 + \frac{t}{n}(\sigma_{nn}^{\circ\gamma})^2 - 2\sum_{i<j=1}^{n}\sigma_{ii}^{\circ\gamma}\sigma_{jj}^{\circ\gamma}\right]. \quad (5.5)$$

We start with the problem

$$min f_\gamma, \quad \text{subject to} \quad \Gamma: \sigma_{11}^{\circ\gamma} + \sigma_{22}^{\circ\gamma} + \cdots + \sigma_{nn}^{\circ\gamma} = k^\gamma,$$

where k^γ is a real constant.

By comparing Equation (5.5) and Lemma 2.4, it is easy to see that

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$$a = \frac{t}{n} + \frac{b(t)}{n-1}, \quad b = \frac{t}{n}.$$

Hence, a critical point of the problem has the following form:

$$\begin{cases} \sigma_{11}^{\circ\gamma} = \sigma_{22}^{\circ\gamma} = \cdots = \sigma_{n-1n-1}^{\circ\gamma} = \dfrac{1}{\dfrac{t}{n} + \dfrac{b(t)}{n-1} + 1} k^{\gamma} \\ \sigma_{nn}^{\circ\gamma} = \dfrac{1}{\dfrac{t}{n} + 1} k^{\gamma}. \end{cases} \tag{5.6}$$

Thus, we get

$$\mathcal{T} \geq 0,$$

which implies

$$2\tau \leq 2tC^{\circ} + 2b(t)C^{\circ}(L_{n-1}) + \frac{c}{4}\{n^2 - (trJ)^2 + 2 \parallel P \parallel^2\} + \frac{n}{2}(\mathcal{C} + \mathcal{C}^*) - \frac{n^2}{2}(\parallel \mathcal{H} \parallel^2 + \parallel \mathcal{H}^* \parallel^2). \tag{5.7}$$

From this it follows that

$$\begin{aligned} \rho &\leq \frac{2t}{n(n-1)} C^{\circ} + \frac{2b(t)}{n(n-1)} C^{\circ}(L_{n-1}) + \frac{c}{4n(n-1)}\{n^2 - (trJ)^2 + 2 \parallel P \parallel^2\} \\ &\quad + \frac{1}{2(n-1)}(\mathcal{C} + \mathcal{C}^*) - \frac{n}{2(n-1)}(\parallel \mathcal{H} \parallel^2 + \parallel \mathcal{H}^* \parallel^2) \\ &= \frac{2}{n(n-1)} \delta_c^{\circ}(t; n-1) + \frac{c}{4n(n-1)}\{n^2 - (trJ)^2 + 2 \parallel P \parallel^2\} \\ &\quad + \frac{1}{2(n-1)}(\mathcal{C} + \mathcal{C}^*) - \frac{n}{2(n-1)}(\parallel \mathcal{H} \parallel^2 + \parallel \mathcal{H}^* \parallel^2), \end{aligned}$$

which is the required inequality. The equality in Equation (5.1) holds if and only if we have the equality in the all the previous inequalities. Thus, the equality holds in Equation (5.1) if and only if the relations in Equation (5.2) are true.

Remark 5.2 A similar result can also be obtained for generalized normalized δ-Casorati curvatures $\hat{\delta}_c(t; n-1)$ and $\hat{\delta}_c^*(t; n-1)$.

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On Statistical Approximation Properties of (p, q)-Bleimann-Butzer-Hahn Operators

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Abstract: The aim of this paper is to introduce a generalization of the (p, q)-Bleimann-Butzer-Hahn operators based on (p, q)-integers and obtain Korovkin's type statistical approximation theorem for these operators. Also, we establish the rate of convergence of these operators using the modulus of continuity. Furthermore, we introduce (p, q)-Bleimann-Butzer-Hahn bivariate operators and study their rate of convergence.

Keywords: Korovkin Theorem, Rate of Convergence, Bivariate Operators
AMS Subject Classifications (2010): 41A10, 41A25, 41A36, 40A30

1. Introduction and Preliminaries

In order to approximate continuous functions defined on the positive half axis, Bleimann, Butzer and Hahn (BBH) introduced, in 1980, the following linear positive operators in [4];

$$L_n(f;x) = \frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\begin{bmatrix} n \\ k \end{bmatrix} x^k, x \geq 0 \tag{1.1}$$

The advent of q-calculus created a new venue of research in Approximation Theory. Lupas [14] introduced the first q-analogue of the Bernstein polynomials in 1987. Phillips [19] presented another modification of Bernstein polynomials in 1997. He also established results for the convergence and the Voronovskaja's type asymptotic expansion for these operators. The q-analogue of the BBH-type operators in (1.1) is defined as

$$L_n^q(f;x) = \frac{1}{l_n(x)}\sum_{k=0}^{n} f\left(\frac{[k]_q}{[n-k+1]_q q^k}\right) q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k, \tag{1.2}$$

where $l_n(x) = \prod_{k=0}^{n-1} (1 + q^s x)$.

In recent decades, the concept of (p, q)-calculus has also been introduced. Many researchers have used (p, q)-calculus to establish new and interesting results in Approximation Theory. Recently, Mursaleen et al [16] introduced the first (p, q) analogue of Bernstein operators and (p, q)-analogue of Bernstein-Stancu operators in [17]. They have investigated the

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approximation and convergence properties of these operators.

Let us give rudiments of (p,q)-calculus. The (p,q)-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q},\quad n=0,1,2,\ldots,0<q<p\leq 1,$$

whereas q-integers are given by

$$[n]_q=\frac{1-q^n}{1-q},\quad n=0,1,2,\ldots,0<q<1,$$

for $n\in\mathbb{N}$. It is clear that the two concepts are different and the former is a generalization of the latter.
Also, we have (p,q)-binomial expansion as follows

$$(ax+by)_{p,q}^n:=\sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_{p,q} a^{n-k}b^k x^{n-k}y^k,$$
$$(x+y)_{p,q}^n=(x+y)(px+qy)(p^2x+q^2y)\ldots(p^{n-1}x+q^{n-1}y),$$
$$(1-x)_{p,q}^n=(1-x)(p-qx)(p^2-q^2x)\ldots(p^{n-1}-q^{n-1}x)$$

and the (p,q)-binomial coefficients are defined by

$$\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}!\,[n-k]_{p,q}!}.$$

By simple calculation, we have the following relation

$$q^k[n-k+1]_{p,q}=[n+1]_{p,q}-p^{n-k+1}[k]_{p,q}.$$

For details on q-calculus and (p,q)-calculus, one is referred to [22] and [10, 20] respectively.
The concept of statistical convergence was introduced by Fast [8] in the circa 1950 and in recent times it has become an active area of research. The concept of the limit of sequence has been generalized to a statistical limit through the natural density of a set K of positive integers, defined as

$$\delta(K)=\lim_{n\to\infty}\frac{1}{n}\{k\leq n \quad for \quad k\in K\}$$

provided this limit exists [18]. We say that the sequence $x=(x_n)$ statistically converges to a number l if for each $\epsilon>0$, the density of the set $\{k:|x_k-l|\geq\epsilon\}$ is zero. We denote it by $st-\lim_k x_k=l$. It is easily seen that every convergent sequence is statistically convergent but not inversely.

The main purpose of this paper is to introduce a modification of the operators defined by

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Mursaleen et al. [15] and investigate statistical approximation properties of the operators with the aid of Korovkin type theorem (see [11, 12]) and estimate the rate of their statistical convergence.
Based on (p,q)-integers, we construct (p,q)-analogue of BBH operators and we call them as (p,q)-Bleimann-Butzer-Hann operators and investigate their Korovkin's type statistical approximation properties by using the test functions $(\frac{t}{1+t})^{\nu}$ for $\nu = 0,1,2$. Also, for space of generalized Lipschitz-type maximal functions we give a pointwise estimation.

Let $C_B(\mathbb{R}_+)$be the set of all bounded and continuous functions on $\mathbb{R}_+$. Then $C_B(\mathbb{R}_+)$ is a linear normed space with the norm

$$\parallel f \parallel_{C_B} = \sup_{x\in R_+} |f(x)|.$$

Let ω denote the modulus of continuity which satisfies the following:
(1) ω is a non-negative inceasing function on $\mathbb{R}_+$
(2) $\omega(\delta_1+\delta_2) \leq \omega(\delta_1)+\omega(\delta_2)$
(3) $\lim_{\delta\to 0} \omega(\delta) = 0.$

Let H_ω be the space of all real-valued funtions f defined on the semi-axis $\mathbb{R}_+$ satisfying the condition

$$|f(x)-f(y)| \leq \omega(|\frac{x}{1+x} - \frac{y}{1+y}|),$$

for any $x, y \in \mathbb{R}_+$.

We state the following theorem from [9].

Theorem 1.1 Let $\{A_n\}$ be the sequence of positive linear operators from H_ω into $C_B(\mathbb{R}_+)$, satisfying the conditions

$$\lim_{n\to\infty} ||A_n((\frac{t}{1+t})^{\nu};x) - (\frac{x}{1+x})^{\nu}||_{C_B},$$

for ν =0,1,2. Then for any function $f \in H_\omega$

$$\lim_{n\to\infty} ||A_n(f) - f||_{C_B} = 0.$$

Now we introduce (p,q)-Bleimann-Butzer-Hahn type operators based on (p,q)-integers as follows:

$$L_n^{p,q}(f;x) = \frac{pq}{\ell_n^{p,q}(x)} \sum_{k=0}^{n} f(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}) p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \tag{1.3}$$

where $x \geq 0, \quad 0 < q < p \leq 1,$ and

$$\ell_n^{p,q}(x) = \prod_{s=0}^{n-1} (p^s + q^s x),$$

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and f is defined on the semi-axis $\mathbb{R}_+$. Also, by induction, we construct the Euler's identity based on (p,q)-integer as

$$\prod_{s=0}^{n-1}(p^s+q^sx)=\sum_{k=0}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k. \tag{1.4}$$

If we put $p=1$, we obtain q-BBH operators in [3]. In (1.3), if we take $f\left(\frac{[k]_{p,q}}{[n-k+1]_{p,q}}\right)$ in place of $f\left(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)$, then we obtain the usual generalization of Bleimann, Butzer and Hahn operators based on (p,q)-integers and then it is not possible to obtain explicit expressions for the monomials t^ν and $\left(\frac{t}{1+t}\right)^\nu$ for $\nu=0,1,2$. Explicit formulas for the monomials $\left(\frac{t}{1+t}\right)^\nu$ for $\nu=0,1,2$ are obtainable only if we define the Bleimann, Butzer and Hahn operators as in (1.3). It is to note that these operators are more flexible than the classical BBH operators and q-BBH operators. That is depending on the selection of (p,q) −integers, the rate of convergence of $(p,\ q)$-BBH operators is as good as the classical one at least.

2. Main Results

We need the following lemma.

Lemma **2.1** Let $L_n^{p,q}(.;x)$ be given by (1.3). Then for any $x\geq 0$ and $0<q<p\leq 1$, we have the following identities

(1) $L_n^{p,q}(1;x)=pq$,

(2) $L_n^{p,q}(\frac{t}{1+t};x)=\frac{p^2q[n]_{p,q}}{[n+1]_{p,q}}(\frac{x}{1+x})$,

(3) $L_n^{p,q}((\frac{t}{1+t})^2;x)=\frac{p^2q^3[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2}\frac{x^2}{(1+x)(p+qx)}+\frac{p^{n+2}q[n]_{p,q}}{[n+1]_{p,q}^2}(\frac{x}{1+x})$.

Proof. (1) We have

$$L_n^{p,q}(1;x)=\frac{pq}{\ell_n^{p,q}(x)}\sum_{k=1}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k.$$

But for $0<q<p\leq 1$, we have

$$\sum_{k=1}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k=\prod_{s=0}^{n-1}(p^s+q^sx)=\ell_n^{p,q}(x),$$

and so

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$$L_n^{p,q}(1;x) = \frac{pq}{l_n^{p,q}(x)} \times l_n^{p,q}(x) = pq,$$

and this proves (1).

(2) Let

$$t = \frac{p^{n-k+1[k]_{p,q}}}{[n-k+1]_{p,q}q^k},$$

then

$$\frac{t}{1+t} = \frac{[k]_{p,q}p^{n+1-k}}{[n+1]p,q}$$

$$\begin{aligned}
L_n^{p,q}(\frac{t}{1+x};x) &= \frac{pq}{\ell_n^{p,q}(x)}\sum_{k=1}^{n}\frac{[k]_{p,q}p^{n-k+1}}{[n+1]_{p,q}}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k \\
&= \frac{pq}{\ell_n^{p,q}(x)}\sum_{k=1}^{n}\frac{[n]_{p,q}p^{n-k+1}}{[n+1]_{p,q}}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n-1\\k-1\end{bmatrix}_{p,q}x^k \\
&= x(\frac{pq}{\ell_n^{p,q}(x)}.\frac{[n]_{p,q}}{[n+1]_{p,q}}p)\sum_{k=1}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n-1\\k\end{bmatrix}_{p,q}(qx)^k \\
&= p^2q\frac{[n]_{p,q}}{[n+1]_{p,q}}\frac{x}{1+x},
\end{aligned}$$

which proves (2).

(3) $L_n^{p,q}\left(\frac{t^2}{(1+t)^2};x\right) = \frac{pq}{\ell_n^{p,q}(x)}\sum_{k=1}^{n}\frac{[k]_{p,q}^2p^{2(n-k+1)}}{[n+1]_{p,q}^2}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k.$

Now we have

$$[k]_{p,q} = p^{k-1} + q[k-1]_{p,q}, \text{and } [k]_{p,q}^2 = q[k]_{p,q}[k-1]_{p,q} + p^{k-1}[k]_{p,q},$$

using it in above, we get

$$\begin{aligned}
&L_n^{p,q}\left(\frac{t^2}{(1+x)^2};x\right) \\
&= \frac{pq}{\ell_n^{p,q}(x)}\sum_{k=1}^{n}\frac{q[k]_{p,q}[k-1]_{p,q}p^{2(n-k+1)}}{[n+1]_{p,q}^2}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k \\
&\quad + \frac{pq}{\ell_n^{p,q}(x)}\sum_{k=1}^{n}p^{k-1}\frac{[k]_{p,q}p^{2n-2k+2}}{[n+1]_{p,q}^2}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k
\end{aligned}$$

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$$= \frac{pq}{\ell_n^{p,q}(x)} \frac{q[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=1}^{n} p^{(2n-2k-2)+\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{p,q} x^k$$

$$+ \frac{pq}{\ell_n^{p,q}(x)} \frac{[n]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=1}^{n} p^{(k-1)+(2n-2k-2)+\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} x^k$$

$$= x^2 \frac{pq}{\ell_n^{p,q}(x)} \frac{q[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=1}^{n-2} p^{((2n-2k-2)+\frac{(n-k-2)(n-k-3)}{2}} q^{\frac{(k+1)(k+2)}{2}} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} x^k$$

$$+ x \frac{pq}{\ell_n^{p,q}(x)} \frac{[n]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=1}^{n-1} p^{(k+(2n-2k)+\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} x^k$$

$$= x^2 \frac{pq}{\ell_n^{p,q}(x)} \frac{pq^2[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=1}^{n-2} p^{(n-k)(n-k-1)} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} (q^2x)^k$$

$$+ x \frac{pq}{\ell_n^{p,q}(x)} \frac{p^{n+1}[n]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=1}^{n-1} p^{(n-k)(n-k-1)} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} (qx)^k$$

$$= \frac{p^2q^3[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \frac{x^2}{(1+x)(p+qx)} + \frac{p^{n+2}q[n]_{p,q}}{[n+1]_{p,q}^2} (\frac{x}{1+x}).$$

This proves (3). □

Korovkin's Type Approximation

We obtain the Korovkin's type statistical approximation properties (see [1]) for the operators defined by (1.3), using Theorem 1.1. In order to obtain the convergence reesults for the operators $L_n^{p,q}(.;x)$, we take $q = q_n$, $p = p_n$ where $q_n \in (0,1)$ and $p_n \in (q_n, 1]$ that satisfy

$$\lim_{n} \quad p_n = 1, \quad \lim_{n} \quad q_n = 1. \qquad (2.1)$$

Theorem 2.2 Let $p = (p_n)$ and $q = (q_n)$ satisty (2.1) for $0 < q_n < p_n \leq 1$, and if $L_n^{p_n,q_n}$ is defined by (1.3), then for any function $f \in H_w$, one obtains

$$st - \lim_{n} ||L_n^{p_n,q_n}(f;x) - f||_{C_B} = 0. \qquad (2.2)$$

Proof. In the light of Theorem 1.1, it is sufficient to prove that

$$st - \lim_{n\to\infty} \quad \| L_n^{p_n,q_n}((\frac{t}{1+t})^\nu; x) - (\frac{x}{1+x})^\nu \|_{C_B} = 0, \quad for \quad \nu = 0,1,2. \qquad (2.3)$$

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From Lemma 2.1, the first condition of (2.2) is easily obtained for $\nu=0$. Also, we can easily see from (2) of Lemma 2.1 (2) that

$$\| L_n^{p_n,q_n}((\frac{t}{1+t})^{\nu};x)-(\frac{x}{1+x})^{\nu}||_{C_B}\leq |\frac{p_nq_n[n]_{p_n,q_n}}{[n+1]_{p_n}q_n}-1|$$
$$=1-p_nq_n\frac{[n]_{p_n},q_n}{[n+1]_{p_n},q_n}.$$

Now for a given $\epsilon>0$, we define the following sets

$$U=\{n:||L_n^{p_n,q_n}(\frac{t}{1+t};x)-\frac{x}{1+x}||\geq\epsilon\},$$
$$U_1=\{n:1-p_nq_n\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\geq\epsilon\}.$$

It is obvious that $U\subset U_1$, so we have

$$\delta\{k\leq n:\| L_n^{p_n,q_n}(\frac{t}{1+t};x)-\frac{x}{1+x}\|\geq\epsilon\}\leq\delta\{k\leq n:1-p_nq_n\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\geq\epsilon\}.$$

Now using (2.1) it is clear that

$$st-\lim_n(1-p_nq_n\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}})=0,$$
$$\delta\{k\leq n:1-p_nq_n\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\geq\epsilon\}=0,$$
$$st-\lim_n||L_n^{p_n,q_n}(\frac{t}{1+t};x)-\frac{x}{1+x}||_{C_B}=0,$$

which proves that the condition (2.2) holds for $\nu=1$. To verify this for $\nu=2$, consider (3) of Lemma 2.1. Then, we see that

$$||L_n^{p_n,q_n}((\frac{t}{1+t})^2;x)-(\frac{x}{1+x})^2||_{C_B}$$
$$=\sup_{x\geq0}\left\{\frac{x^2}{(1+x)^2}\left(\frac{p_n^2q_n^3[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}.\frac{1+x}{p_n+q_nx}-1\right)+\frac{p_n^{n+2}q_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\frac{x}{1+x}\right\}$$

After calculations, we get

$$\frac{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}=\frac{1}{q_n^3}\{1-p_n^n(2+\frac{q_n}{p_n})\frac{1}{[n+1]_{p_n,q_n}}+(p_n^n)^2(1+\frac{q_n}{p_n})\frac{1}{[n+1]_{p_n,q_n^2}}\},$$

and

$$\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}=\frac{1}{q_n}(\frac{1}{[n+1]_{p_n,q_n}}-p_n^n\frac{1}{[n+1]_{p_n,q_n}^2}).$$

Then we are led to

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$$||L_n^{p_n,q_n}((\frac{t}{1+t})^2;x)-(\frac{x}{1+x})^2||_{C_B}\leq |(p_n^2-1)+p_n^{n+2}(\frac{-1}{[n+1]_{p_n,q_n}}+\frac{p_n^n}{[n+1]_{p_n,q_n}^2})$$
$$+p_n^{n+1}(\frac{-q_n}{[n+1]_{p_n,q_n}}+\frac{p_n^2}{[n+1]_{p_n,q_n}}+\frac{p_n^2q_n}{[n+1]_{p_n,q_n}})|.$$
$$=1-p_n^2+p_n^{n+2}(\frac{1}{[n+1]_{p_n,q_n}}-\frac{p_n^n}{[n+1]_{p_n,q_n}^2})$$
$$+p_n^{n+1}(\frac{q_n}{[n+1]_{p_n,q_n}}-\frac{p_n^2+p_n^nq_n}{[n+1]_{p_n,q_n}^2}).$$

Now if we denote

$$1-p_n^2,\qquad p_n^{n+2}(\frac{1}{[n+1]_{p_n,q_n}}-\frac{p_n^n}{[n+1]_{p_n,q_n}^2})$$

and

$$p_n^{n+1}\left(\frac{q_n}{[n+1]_{p_n,q_n}}-\frac{p_n^2+p_n^nq_n}{[n+1]_{p_n,q_n}^2}\right)$$

by α_n,β_n and γ_n respectively, then using (2.1), we find that

$$st-\lim_n \alpha_n=0,\quad st-\lim_n \beta_n=0,\quad st-\lim_n \gamma_n=0. \tag{2.4}$$

Now for a given $\epsilon>0$, we define the following sets

$U=\{n: ||L_n^{p_n,q_n}((\frac{t}{1+t})^2\,;x)\ -\ (\frac{x}{1+x})^2||_{C_B}\geq\epsilon\}$,

$U_1=\left\{n:\alpha_n\geq\frac{\epsilon}{3}\right\}$,

$U_2=\left\{n:\beta_n\geq\frac{\epsilon}{3}\right\}$,

$U_3=\left\{n:\gamma_n\geq\frac{\epsilon}{3}\right\}$.

It is obvious that $U\subseteq U_1\cup U_2\cup U_3$. So, we have

$$\delta\{k\leq n: ||L_n^{p_n,q_n}(\ (\frac{t}{1+t})^2;x)-\ (\frac{x}{1+x})^2||_{C_B}\geq\epsilon\}$$
$$\leq\delta\{k\leq n:\alpha_n\geq\frac{\epsilon}{3}\}+\delta\{k\leq n:\beta_n\geq\frac{\epsilon}{3}\}+\delta\{k\leq n:\gamma_n\geq\frac{\epsilon}{3}\}.$$

Now by virtue of (2.4), the right-hand side of the above inequality is trivial, so we get

$$st-\lim_n ||L_n^{p_n,q_n}(\ (\frac{t}{1+t})^2;x)-\ (\frac{x}{1+x})^2||_{C_B}=0.$$

Hence the proof of the theorem is complete. □

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3. Rate of Convergence

In this section, we calculate the rate of convergence of the operators (1.3) by means of modulus of continuity and Lipschitz type maximal functions.

Let $\delta > 0$. The modulus of continuity for $f \in H_\omega$ is defined by

$$\widetilde{\omega}(f;\delta) = \sum_{\substack{|\frac{t}{1+t}-\frac{x}{1+x}|\leq\delta \\ x,t\geq 0}} |f(t)-f(x)|$$

where $\widetilde{\omega}(f;\delta)$ satisfies the following conditions, for all $f \in H_\omega(\mathbb{R}_+)$,

(1) $\lim_{\delta\to\infty}\widetilde{\omega}(f;\delta) = 0$,

(2) $|f(t)-f(x)| \leq \widetilde{\omega}(f;\delta)(\frac{|\frac{t}{1+t}-\frac{x}{1+x}|}{\delta}+1)$.

Theorem 3.1 Let $p=(p_n)$ and $q=(q_n)$ satisfy (2.1) for $0<q_n<p_n\leq 1$, and let $L_n^{p_nq_n}$ be defined by (1.3). Then for each $x \geq 0$ and for any function $f \in H_\omega$, we have

$$|L_n^{p_nq_n}(f;x)-f| \leq 2\widetilde{\omega}(f;\sqrt{(}\delta_n(x)),$$

where

$$\delta_n(x) = \frac{x^2}{(1+x)^2}(\frac{p_n^2q_n^3[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\frac{1+x}{p_n+q_nx}-2\frac{p_nq_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}+1) + \frac{p^{n+2}q_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\frac{x}{1+x}.$$

Proof.

$$|L_n^{p_nq_n}(f;x)-f| \leq L_n^{p_nq_n}(|f(t)-f(x)|;x) \leq \widetilde{\omega}(f;\delta)\left\{1+\frac{1}{\delta}L_n^{p_nq_n}\left(\left|\frac{t}{1+t}-\frac{x}{1+x}\right|;x\right)\right\}.$$

Now by using the Cauchy-Schwarz inequality, one obtains

$$|L_n^{p_n,q_n}(f;x)-f| \leq \widetilde{\omega}(f;\delta_n)\{1+\frac{1}{\delta_n}[L_n^{p_nq_n}((\frac{t}{1+t}-\frac{x}{1+x})^2;x)]^{\frac{1}{2}}(L_n^{p_nq_n}(1;x))^{\frac{1}{2}}\}$$

$$\leq \widetilde{\omega}(f;\delta_n)\{1+\frac{1}{\delta_n}[\frac{x^2}{(1+x)^2}(\frac{p_n^2q_n^3[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\frac{1+x}{p_n+q_nx} - 2\frac{p_n,q_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}+1)+\frac{p^{n+2}q_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\frac{x}{1+x}]^{\frac{1}{2}}\}.$$

This completes the proof. □

Now we will give an estimate concerning the rate of convergence by means of Lipschitz type maximal functions. In [3], the Lipschitz type maximal function space on $E \subset \mathbb{R}_+$ is defined as

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$$\widetilde{W}_{\alpha,E} = \{f: \sup(1+x)^{\alpha}\tilde{f}_{\alpha}(x) \leq M\frac{1}{(1+y)^{\alpha}}: x \geq 0, \quad \text{and} \quad y \in E\} \tag{3.1}$$

where f is bounded and continuous function on $\mathbb{R}_{+}$, M is a positive constant and $0 < \alpha \leq 1$, and Lipschitz type maximal function are defined as as follows [13]:

$$f_{\alpha}(x,t) = \sum_{\substack{t>0 \\ t\neq 0}} \frac{|f(t)-f(x)|}{|x-t|^{\alpha}}. \tag{3.2}$$

We denote by $d(x,E)$, the distance between x and the set E. That is

$$d(x,E) = inf\{|x-y|; y \in E\}.$$

Theorem 3.2. For all $f \in \widetilde{W}_{\alpha,E}$, we have

$$|L_n^{p_n,q_n}(f;x) - f(x)| \leq M(\delta_n^{\frac{\alpha}{2}}(x)2(d(x,E))^{\alpha}), \tag{3.3}$$

where $\delta_n(x)$ is defined as in Theorem 3.1.

Proof. Let $\overline{E}$ denote the clouser of the set E. Then there exits an $x_0 \in \overline{E}$ such that $|x - x_0| = d(x,E)$, where $x \in \mathbb{R}_{+}$. Thus, we can write

$$|f - f(x)| \leq |f - f(x_0)| + |f(x_0) - f(x)|.$$

Since $L_n^{p_n,q_n}$ are the positive linear operators, so for $f \in \widetilde{W}_{\alpha,E}$, by using the previous inequality, we have

$$\begin{aligned} |L_n^{p_n,q_n}(f;x) - f(x)| &\leq |L_n^{p_n,q_n}(f - f(x_0)|;x) + |f(x_0) - f(x)|L_n^{p_n,q_n}(1;x) \\ &\leq M(L_n^{p_n,q_n}(|\frac{t}{1+t} - \frac{x_0}{1+x_0}|^{\alpha};x) + \frac{|x-x_0|^{\alpha}}{(1+x)^{(}\alpha)(1+x_0)^{\alpha}}L_n^{p_n,q_n}(1;x)) \end{aligned}$$

Then $(a+b)^{\alpha} \leq a^{\alpha} + b^{\alpha}$ consequently implies

$$\begin{aligned} &L_n^{p_n,q_n}(|\frac{t}{1+t} - \frac{x_0}{1+x_0}|^{\alpha};x) \\ &\leq L_n^{p_n,q_n}(|\frac{t}{1+t} - \frac{x}{1+x}|^{\alpha};x) + L_n^{p_n,q_n}(|\frac{x}{1+x} - \frac{x_0}{1+x_0}|^{\alpha};x) \\ &L_n^{p_n,q_n}(|\frac{t}{1+t} - \frac{x_0}{1+x_0}|^{\alpha};x) \\ &\leq L_n^{p_n,q_n}(|\frac{t}{1+t} - \frac{x}{1+x}|^{\alpha};x) + \frac{|x-x_0|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}L_n^{p_n,q_n}(1;x). \end{aligned}$$

By using the Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

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$$L_n^{p_n,q_n}(|\frac{t}{1+t}-\frac{x_0}{1+x_0}|^\alpha;x) \leq L_n^{p_n,q_n}(|\frac{t}{1+t}-\frac{x}{1+x}|^2;x)^{\frac{\alpha}{2}}(L_n^{p_n,q_n}(1;x))^{\frac{2-\alpha}{2}}$$
$$+\frac{|x-x_0|^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}L_n^{p_n,q_n}(1;x)$$
$$=\delta_n^{\frac{\alpha}{2}}(x)+\frac{|x-x_0|^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}L_n^{p_n,q_n}(1;x).$$

This completes the proof. □

Corollary 3.3 If we take $E=\mathbb{R}_+$ as a particular case of Theorem 3.2, then for all $f \in \widetilde{W}_{\alpha,\mathbb{R}_+}$, one infers

$$|L_n^{p_n,q_n}(f;x)-f(x)| \leq M\delta_n^{\frac{\alpha}{2}}(x),$$

where $\delta_n(x)$ is defined as in Theorem 3.1.

Theorem 3.4 If

$$x \in (0,\infty)\backslash\{p^{n-k+1}\frac{[k]_{p,q}}{[n-k+1]_{p,q}q^k}: k=0,1,2,\dots,n\},$$

then

$$L_n^{p,q}(f;x)-f(\frac{px}{q}) = -\frac{x^{n+1}}{l_n^{p,q}(x)}[\frac{px}{q};\frac{p[n]_{p,q}}{q^n};f]pq^{\frac{n(n-1)}{2}-n}$$
$$+\frac{x}{l_n^{p,q}(x)}\sum_{k=0}^{n-1}[\frac{px}{q};p^{n-k+1}\frac{[k]_{p,q}}{[n-k+1]_{p,q}q^k};f]$$
$$\times\frac{1}{[n-k]^{p,q}}p^{\frac{(n-k)(n-k-1)}{2}-(k-n)-1}q^{\frac{k(k-1)}{2}-k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^n.$$

Proof. We have

$$L_n^{p,q}(f;x)-f\left(\frac{px}{q}\right)=\frac{pq}{l_n^{p,q}(x)}\sum_{k=0}^{n}\left\{f\left(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)-f\left(\frac{px}{q}\right)\right\}$$
$$\times p^{\frac{(n-k)(n-k-1)}{2}+1}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k$$
$$=-\frac{1}{l_n^{p,q}(x)}\sum_{k=0}^{n}\left(\frac{px}{q}-\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)\left[\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]$$
$$\times p^{\frac{(n-k)(n-k-1)}{2}+1}q^{\frac{k(k-1)}{2}+1}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k$$

By using $\frac{[k]_{p,q}}{[n-k+1]_{p,q}}\begin{bmatrix}n\\k\end{bmatrix}=\begin{bmatrix}n\\k-1\end{bmatrix}_{p,q}$ we obtain

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$$L_n^{p,q}(f;x)-f(\frac{px}{q})$$
$$=-\frac{x}{l_n^{p,q}(x)}\sum_{k=0}^{n}[\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f]p^{\frac{(n-k)(n-k-1)}{2}+2}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k$$
$$+\frac{1}{l_n^{p,q}(x)}\sum_{k=1}^{n}[\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f]p^{\frac{(n-k)(n-k-1)}{2}-(k-n-1)-1}q^{\frac{k(k-1)}{2}-(k-1)}\times\begin{bmatrix}n\\k-1\end{bmatrix}_{p,q}x^k$$

$$=-\frac{x}{l_n^{p,q}(x)}\sum_{k=0}^{n}[\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f]p^{\frac{(n-k)(n-k-1)}{2}+2}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k$$
$$+\frac{x}{l_n^{p,q}(x)}\sum_{k=0}^{n-1}[\frac{px}{q};\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}};f]p^{\frac{(n-k)(n-k-1)}{2}-(k-n-1)-2}q^{\frac{k(k-1)}{2}-k}\times\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k$$

$$=-\frac{x^{n+1}}{l_n^{p,q}(x)}\sum_{k=1}^{n}[\frac{px}{q};\frac{p[n]_{p,q}}{q^n};f]pq^{\frac{n(n-1)}{2}-n}+\frac{x}{l_n^{p,q}(x)}\sum_{k=0}^{n-1}\{[\frac{px}{q};\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^n};f]$$
$$-[\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f]\}p^{\frac{(n-k)(n-k-1)}{2}-(k-n-1)-2}q^{\frac{k(k-1)}{2}-k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k.$$

On using the results
$$\left[\frac{px}{q};\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}};f\right]-\left[\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]$$
$$=\left(\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}}-\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)\left\{\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}};f\right\},$$
and
$$\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}}-\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}=[n+1]_{p,q},$$
we arrive at the desired result.
Hence the theorem is proved. □

4. Some Generalisation of $L_n^{p,q}$

In this part, we present some generalizations of the operators $L_n^{p,q}$ based on (p,q)-integers similar to work done in [3, 5]. We consider a sequence of linear positive operators based on (p,q)-integers as follows:

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$$L_n^{(p,q),\gamma}(f;x) = \frac{pq}{l_n^{p,q}(x)}\sum_{k=0}^{n}\left(\frac{p^{n-k+1}[k]_{p,q}+\gamma}{b_n,k}\right)p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k,(\gamma\in\mathbb{R}) \tag{4.1}$$

where $b_{n,k}$ satisfy the following

$$p^{n-k+1}[k]_{p,q}+b_{n,k}=c_n \quad and \quad \frac{[n]_{p,q}}{c_n}\to 1 \quad \text{for} \quad n\to\infty.$$

It is easy to check that if $b_{n,k}=q^k[n-k+1]_{p,q}+\beta$ for any n,k and $0<q<p\leq 1$, then $c_n=[n+1]_{p,q}+\beta$. If we choose $p=1$, then the operators reduce to the generalization of q-BBH operators in [3] which turn out to be Stancu-type generalization of Bleimann, Butzer, and Hahn operators based on q-integers in [21]. If we choose $\gamma=0,\ q=1$ as in [3] for $p=1$, the operators become a special case of the Balzs-type generalization of the q-BBH operators [3] given in [5]. We derive the following theorem.

Theorem 4.1 Let $p=(p_n)$ and $q=(q_n)$ satisfy (2.1) for $0<q_n<p_n\leq 1$ and let $L_n^{(p,q),\gamma}$ be defined by (4.1). Then for any function $f\in\widetilde{W}_\alpha,[0,\infty)$, one has

$$\lim_n ||L_n^{(p_n,q_n),\gamma}(f;x)-f(x)||_{CB}\leq 3M \times max\{(\frac{[n]_{p_n,q_n}}{c_n+\gamma})^\alpha(\frac{\gamma}{[n]_{p_n,q_n}})^\alpha, |1-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}|^\alpha \quad (\frac{p_nq_n[n]_{p_nq_n}}{[n+1]_{p_n,q_n}})^\alpha, 1-2\frac{p_nq_n[n]_{p_nq_n}}{[n+1]_{p_n,q_n}} + \frac{p_nq_n[n]_{p_nq_n}[n-1]p_n,q_n}{[n+1]^2_{p_n,q_n}}\}.$$

Proof. Using (1.3) and (4.1), we have

$$\| L_n^{(p_n,q_n),\gamma}(f;x)-f(x)\|_{C_B} \leq \frac{pq}{l_n^{p,q}(x)}\sum_{k=0}^{n}|f(\frac{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma}{b_n,k}) - f(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_n,k})|p_n^{\frac{(n-k)(n-k-1)}{2}}q_n^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p_n,q_n}x^k + \frac{pq}{l_n^{p,q}(x)}\sum_{k=0}^{n}|f(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_n,k}) - f(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{[n-k+1]_{p_n,q_n}q_n^n})|p_n^{\frac{(n-k)(n-k-1)}{2}}q_n^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p_n,q_n}x^k,$$

we have

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$$\begin{aligned}
&\parallel L_n^{(p_n,q_n),\gamma}(f;x)-f(x)\parallel_{C_B}\\
&\leq \frac{pq}{l_n^{p,q}(x)}\sum_{k=0}^{n}|f(\frac{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma}{b_n,k}\\
&\qquad -f(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_n,k})|p_n^{\frac{(n-k)(n-k-1)}{2}}q_n^{\frac{k(k-1)}{2}+1}\begin{bmatrix}n\\k\end{bmatrix}_{p_n,q_n}x^k\\
&+\frac{pq}{l_n^{p,q}(x)}\sum_{k=0}^{n}|f(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_n,k}\\
&\qquad -f(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{[n-k+1]_{p_n,q_n}q_n^k})|p_n^{\frac{(n-k)(n-k-1)}{2}}q_n^{\frac{k(k-1)}{2}+1}\begin{bmatrix}n\\k\end{bmatrix}_{p_n,q_n}x^k\\
&+|L_n^{(p_n,q_n),\gamma}(f;x)-f(x)|.
\end{aligned}$$

Now for $f\in\widetilde{W}_\alpha,[0,\infty)$, by using Corollary 3.3, we can write

$$\begin{aligned}
&\parallel L_n^{(p_n,q_n),\gamma}(f;x)-f(x)\parallel_{C_B}\\
&\leq \frac{M}{l_n^{p_n,q_n}(x)}\sum_{k=0}^{n}|\frac{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma}{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma+b_{n,k}}\\
&\qquad -f(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+p_n^{n-k+1}[k]_{p_n,q_n+b_{n,k}}q_n^k})|^\alpha p_n^{\frac{(n-k)(n-k-1)}{2}+1}q_n^{\frac{k(k-1)}{2}+1}\\
&\begin{bmatrix}n\\k\end{bmatrix}_{p_n,q_n}x^k+\frac{M}{l_n^{p_n,q_n}(x)}\sum_{k=0}^{n}|\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma+b_{n,k}}\\
&\qquad -\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{p_n^{n-k+1}[k]_{p_n,q_n}+[n-k+1]_{p_n,q_n}q_n^k}|\\
&\qquad\quad \times p_n^{\frac{(n-k)(n-k-1)}{2}+1}q_n^{\frac{k(k-1)}{2}+1}\begin{bmatrix}n\\k\end{bmatrix}_{p_n,q_n}x^k+M\delta_n^{\frac{\alpha}{2}}(x).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\parallel L_n^{(p_n,q_n),\gamma}(f;x)-f(x)\parallel_{C_B}\leq M(\frac{[n]_{p_n,q_n}}{c_n+\gamma})^\alpha(\frac{\gamma}{[n]_{p_n,q_n}})^\alpha\\
&+\frac{M}{l_n^{p_n,q_n}(x)}|1\\
&-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}|^\alpha\sum_{k=0}^{n}(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{[n+1]_{p_n,q_n}})^\alpha p_n^{\frac{(n-k)(n-k-1)}{2}+1}q_n^{\frac{k(k-1)}{2}+1}\begin{bmatrix}n\\k\end{bmatrix}_{p_n,q_n}x^k+M\delta_n^{\frac{\alpha}{2}}(x)\\
&=M(\frac{[n]_{p_n,q_n}}{c_n+\gamma})^\alpha(\frac{\gamma}{[n]_{p_n,q_n}})^\alpha+M|1-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}|^\alpha L_n^{p_n,q_n}((\frac{t}{t+1})^\alpha;x)+M\delta_n^{\frac{\alpha}{2}}(x).
\end{aligned}$$

Using the Hölder's inequality for $p=\frac{1}{\alpha}$ and $q=\frac{1}{1-\alpha}$, we get

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$$\| L_n^{(p_n,q_n),\gamma}(f;x) - f(x) \|_{C_B}$$
$$\leq M(\frac{[n]_{p_n,q_n}}{c_n+\gamma})^\alpha(\frac{\gamma}{[n]_{p_n,q_n}})^\alpha + M|1-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}|^\alpha L_n^{p_n,q_n}((\frac{t}{t+1}); x)^\alpha(L_n^{p_n,q_n}(1;x))^{1-\alpha}$$
$$+M\delta_n^{\frac{\alpha}{2}}(x)$$
$$\leq M(\frac{[n]_{p_n,q_n}}{c_n+\gamma})^\alpha(\frac{\gamma}{[n]_{p_n,q_n}})^\alpha + M|1-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}|^\alpha(\frac{p_nq_n[n]_{p_nq_n}}{[n+1]_{p_n,q_n}}\frac{x}{1+x})^\alpha + M\delta_n^{\frac{\alpha}{2}}(x),$$

which completes the proof. □

5. Construction of the Bivariate Operators

In what follows we construct the bivariate extension of the operators defined in (1.3). We will introduce the statistical convergence and investigate the statistical rate of convergence of these operators.

Let $\mathbb{R}_+^2 = [0,\infty)\times[0,\infty), f:\mathbb{R}_+^2 \to \mathbb{R}$ and $0 < p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2} \leq 1$. Then we define the bivariate companion of the operators (1.3) as follows:

$$L_{n_1,n_2}(f; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y)$$
$$= \frac{p_{n_1}q_{n_1}p_{n_2}q_{n_2}}{l_{n_1}^{p_{n_1},q_{n_1}} \times l_{n_2}^{p_{n_2},q_{n_2}}} \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} f\left(\frac{p_{n_1}^{n_1-k_1}+1[k_1]_{p_{n_1},q_{n_1}}}{[n_1-k_1+1]_{p_{n_1},q_{n_1}} q_{n_1}^{k_1}}, \frac{p_{n_2}^{n_2-k_2}+1[k_2]_{p_{n_2},q_{n_2}}}{[n_2-k_2+1]_{p_{n_2},q_{n_2}} q_{n_2}^{k_2}}\right)$$
$$(p_{n_1}^{\frac{(n_1-k_1)(n_1-k_1-1)}{2}} p_{n_2}^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}} q_{n_1}^{\frac{k_1(k_1-1)}{2}} q_{n_2}^{\frac{k_2(k_2-1)}{2}}) \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{p_{n_1},q_{n_1}} \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{p_{n_2}q_{n_2}} x^{k_1}y^{k_2} \tag{5.1}$$

where

$$l_{n_1}^{p_{n_1},q_{n_1}} = \prod_{s=0}^{n_1-1}(p_{n_1}^s + q_{n_1}^s x)$$

and

$$l_{n_2}^{p_{n_2},q_{n_2}} = \prod_{s=0}^{n_2-1}(p_{n_2}^s + q_{n_2}^s y).$$

Let $K = [0,\infty)\times[0,\infty)$, and $\delta_1, \delta_2 > 0$. The modulus of continuity for the bivariate case is defined as

$\omega_2(g;\delta_1,\delta_2) = \sup\{|g(u_1,v_1) - g(u_2,v_2): (u_1,v_1),(u_2,v_2) \in K$

and $|u_1,u_2| \leq \delta_1, |v_1,v_2| \leq \delta_2\}$, where, for each $g \in H_{\omega_2}; \omega_2(g;\delta_1,\delta_2)$ satisfies

$$|g(u_1,v_1) - g(g(u_2,v_2)| \leq \omega_2\left(g\left|\frac{u_1}{1+u_1} - \frac{u_2}{1+u_2}\right|, \left|\frac{v_1}{1+v_1} - \frac{v_2}{1+v_2}\right|\right).$$

For detailed study of modulus of continuity for the bivariate analogue one is referred to [2]. The first Korovkin type theorem for the statistical approximation for the bivariate analogue

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of linear positive operators defined in the space H_{ω_2} was obtained by Erkus and Duman [6] which is stated below.

Theorem 5.1 Let $\{L_n\}$ be a sequence of positive linear operators from H_{ω_2} into $C_B(K)$. Then, for each $g \in H_{\omega_2}$,

$$st-\lim_n ||L_n(g)-g||=0$$

holds if the following is satisfied

$$st-\lim_n \left\|L_n(g_j)-g_j\right\|=0, \quad \text{for } j=0,1,2,3,$$

where

$$g_0(u,v)=0, \quad g_1(u,v)=\frac{u}{1+u}, \quad g_2(u,v)=\frac{v}{1+v}, \quad g_3(u,v)=(\frac{u}{1+u})^2+(\frac{v}{1+v})^2.$$

To study the statistical convergence of the bivariate operators, the following lemma is essential.

Lemma 5.2 For the bivariate operators defined by (5.1) the followings are obtained:

(1) $L_{n_1,n_2}(f_0;p_{n_1},q_{n_1};p_{n_2},q_{n_2};x,y)=p_{n_1}q_{n_1}p_{n_2}q_{n_2}$,

(2) $L_{n_1,n_2}(f_1;p_{n_1},q_{n_1};p_{n_2},q_{n_2};x,y)=p_{n_1}q_{n_1}p_{n_2}q_{n_2}\dfrac{[n_1]_{p_{n_1},q_{n_1}}}{[n_1+1]_{p_{n_1},q_{n_1}}}\dfrac{x}{1+x}$,

(3) $L_{n_1,n_2}(f_2;p_{n_1},q_{n_1};p_{n_2},q_{n_2};x,y)=p_{n_1}q_{n_1}p_{n_2}q_{n_2}\dfrac{[n_2]_{p_{n_2},q_{n_2}}}{[n_2+1]_{p_{n_2},q_{n_2}}}\dfrac{y}{1+y}$,

(4) $L_{n_1,n_2}(f_3;p_{n_1},q_{n_1};p_{n_2},q_{n_2};x,y)$

$$=p_{n_1}^3q_{n_1}^3p_{n_2}q_{n_2}\frac{[n_1]_{p_{n_1},q_{n_1}}[n_1-1]_{p_{n_1},q_{n_1}}}{[n_1+1]^2_{p_{n_1},q_{n_1}}}\frac{x^2}{(1+x)(p_{n_1}+q_{n_1}x)}$$
$$+p_{n_1}q_{n_1}p_{n_2}q_{n_2}\frac{[n_1]_{p_{n_1}},q_{n_1}}{[n_1+1]_{p_{n_1}},q_{n_1}}\frac{x}{1+x}+p_{n_1}q_{n_1}p_{n_2}^3q_{n_2}^3\frac{[n_2]_{p_{n_2},q_{n_2}}[n_2-1]_{p_{n_2},q_{n_2}}}{[n_2+1]^2_{p_{n_2},q_{n_2}}}$$
$$\times\frac{x^2}{(1+x)(p_{n_2}+q_{n_2}x)}+p_{n_2}q_{n_2}p_{n_2}q_{n_2}\frac{[n_2]_{p_{n_2},q_{n_2}}}{[n_2+1]_{p_{n_2},q_{n_2}}}\frac{y}{1+y}.$$

Proof. Exploiting the proofs for the bivariate operators in [7], the above are easily established. So, we skip the proof.
Now let the sequences

$$p=(p_{n_1}), \quad p=(p_{n_2}), \quad q=(q_{n_1}), \quad q=(q_{n_2})$$

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be statistically convergent to unity but not convergent in usual sense, so we can write them for $0 < p_{n_1}, q_{n_1}, p_{n_2}, q_{n_2} \leq 1$ as

$$st - \lim_{n_1} p_{n_1} = st - \lim_{n_1} q_{n_1} = st - \lim_{n_2} p_{n_2} = st - \lim_{n_2} q_{n_2} = 1. \quad (5.2)$$

Making use of the proof of Theorem 2.2 and conditions (5.2), we establish the statistical convergence of the bivariate operators introduced above.

Theorem 5.3 Let $p = (p_{n_1}), p = (p_{n_2}), q = (q_{n_1})$ and $q = (q_{n_2})$ be the sequences subject to conditions (2.1) and let L_{n_1,n_2} be the sequence of linear positive operators from $H_{\omega_2}(\mathbb{R}_+^2)$ into $C_B(\mathbb{R})$. Then for each $g \in H_{\omega_2}$,

$$st - \lim_{n_1,n_2} ||L_{n_1,n_2}(g) - g|| = 0.$$

Proof. With the aid of the Lemma 5.2, a proof similar to the proof of the Theorem 2.2 can be easily obtained. So, it will be omitted.

Rates of convergence of the bivariate operators

For any $g \in H_{\omega_2}(\mathbb{R}_+^2)$, the modulus of continuity of the bivariate analogue is defined as;

$$\widetilde{\omega}(g;\delta_1,\delta_2) = \sup_{x_1,x_2\geq 0} \{|g(x_1,y_1) - g(x_2,y_2)| : |\frac{x_1}{1+x_1} - \frac{x_2}{1+x_2}| \leq \delta_1,$$
$$|\frac{y_1}{1+y_1} - \frac{y_2}{1+y_2}| \leq \delta_2, (x_1,y_1),(x_1,y_1) \in H_{\omega_2}(\mathbb{R}_+^2)\},$$

where $\delta_1, \delta_2 > 0.$ For details of this sort of modulus, one is referred to [2].

Two chief properties of $\widetilde{\omega}(g;\delta_1,\delta_2)$ are the following

(1) $\widetilde{\omega}(g;\delta_1,\delta_2) \to 0$ as $\delta_1 \to 0$ and $\delta_2 \to 0$ and

(2) $|g(x_1,y_1) - g(x_2,y_2)|$

$$\leq \widetilde{\omega}(g;\delta_1,\delta_2)\left(1 + \frac{\left|\frac{x_1}{1+x_1} - \frac{x_2}{1+x_2}\right|}{\delta_1}\right)\left(1 + \frac{\left|\frac{y_1}{1+y_1} - \frac{2}{1+y_2}\right|}{\delta_2}\right).$$

The next theorem studies the rate of statistical convergence of the bivariate operators defined in (5.1) through the modulus of continuity in $H_{\omega_2}(\mathbb{R}_+^2)$.

Theorem 5.4 Let $p = (p_{n_1}), p = (p_{n_2}), q = (q_{n_1}), q = (q_{n_2})$ be the four sequences obeying conditions of (5.2). Then we have

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y) - f(x,y)| \leq 4p_{n_1}^2 p_{n_2}^2 q_{n_1}^2 q_{n_2}^2 \omega(f;\sqrt{\delta_{n_1}}(x),\sqrt{\delta_{n_2}}(x)),$$

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where

$$\delta_{n_1}(x) = \frac{x^2}{(1+x)^2}(p_{n_1}^2 q_{n_1}^2 \frac{(1+x)}{p_{n_1}+q_{n_1}x}\frac{[n_1]_{p_{n_1},q_{n_1}}[n_1-1]_{p_{n_1},q_{n_1}}}{[n_1^2+1]_{p_{n_1},q_{n_1}}^2} - 2\frac{[n_1]_{p_{n_1},q_{n_1}}}{[n_1+1]_{p_{n_1},q_{n_1}}} + 1)$$
$$+\frac{x}{1+x}\frac{[n_1]_{p_{n_1},q_{n_1}}}{[n_1+1]_{p_{n_1},q_{n_1}}},$$
$$\delta_{n_2}(y) = \frac{y^2}{(1+y)^2}(p_{n_2}^2 q_{n_2}^2 \frac{(1+y)}{p_{n_2}+q_{n_2}y}\frac{[n_2]_{p_{n_2},q_{n_2}}[n_2-1]_{p_{n_2},q_{n_2}}}{[n_2^2+1]_{p_{n_2},q_{n_2}}^2} - 2\frac{[n_2]_{p_{n_2},q_{n_2}}}{[n_2+1]_{p_{n_2},q_{n_2}}}$$
$$+1) + \frac{y}{1+y}\frac{[n_2]_{p_{n_2},q_{n_2}}}{[n_2+1]_{p_{n_2},q_{n_2}}}.$$

Proof. Using the property of the modulus above, we have

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y) - f(x,y)|$$
$$\leq \omega(f:;\delta_{n_1},\delta_{n_2})\{L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$+\frac{1}{\delta_{n_1}}L_{n_1,n_2}(|\frac{t}{1+t}-\frac{x}{1+x}|;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)\}\{L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$+\frac{1}{\delta_{n_2}}L_{n_1,n_2}(|\frac{s}{1+s}-\frac{y}{1+y}|;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)\}.$$

Appling the Cauchy-Schwarz inequality, we get

$$L_{n_1,n_2}(|\frac{t}{1+t}-\frac{x}{1+x}|;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$\leq (L_{n_1,n_2}((\frac{t}{1+t}-\frac{x}{1+x})^2;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y))^{\frac{1}{2}}$$
$$\times (L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)^{\frac{1}{2}}.$$

On substituting this in the above inequality, we get the proof of the theorem.

In the ensuing we shall study the statistical convergence of the bivariate operators using Lipschitz type maximal functions. The Lipschitz type maximal function space on $E \times E \subset \mathbb{R}_+ \times \mathbb{R}_+$ is defined by

$$\widetilde{W}_{\alpha_1,\alpha_2,E^2} = \{f: sup(1+t)^{\alpha_1}(1+s)^{\alpha_2}\tilde{f}_{\alpha_1,\alpha_2}(x,y)$$
$$\leq M\frac{1}{(1+x)^{\alpha_1}}\frac{1}{(1+y)^{\alpha_2}}; x,y \geq 0, (t,s) \in E^2\}. \quad (5.3)$$

where f is a bounded and continuous function on $\mathbb{R}_+$, M is a positive constant and $0 \leq \alpha_1, \alpha_2 \leq 1$ and $\tilde{f}_{\alpha_1,\alpha_2}(x,y)$ is defined below:

$$\tilde{f}_{\alpha_1,\alpha_2}(x,y) = \sup_{t,s\geq 0}\frac{|f(t,s)-f(x,y)|}{|t-x|^{\alpha_1}|s-y|^{\alpha_2}}.$$

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Theorem 5.5 Let $p=(p_{n_1}),\quad p=(p_{n_2}),\quad q=(q_{n_1}),\quad q=(q_{n_2})$ be the four sequences satisfying conditions of (5.2). Then we have

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)-f(x,y)|$$
$$\leq Mp_{n_1}p_{n_2}q_{n_1}q_{n_2}\{\delta_{n_1}(x)^{\frac{\alpha_1}{2}}\delta_{n_2}(y)^{\frac{\alpha_2}{2}}(p_{n_1}p_{n_2}q_{n_1}q_{n_2})$$
$$+\delta_{n_1}(x)^{\frac{\alpha_1}{2}}d(y,E)^{\alpha_2}+\delta_{n_2}(y)^{\frac{\alpha_2}{2}}d(x,E)^{\alpha_1}2d(x,E)^{\alpha_1}d(y,E)^{\alpha_2}\},$$

where $0\leq\alpha_1,\alpha_2\leq 1$ and $\delta_{n_1}(x),\delta_{n_2}(y)$ are defined as in Theorem 3.1 and $d(x,E)=\inf\{|x-y|:y\in E\}$.

Proof. For $x,y\geq 0$ and $(x_1,y_1)\in E\times E,$ we can write

$$|f(t,s)-f(x,y)|\leq|f(t,s)-f(x_1,y_1)|+|f(x_1,y_1)-f(x,y)|.$$

Applying the operator L_{n_1,n_2} to both sides of the above inequality and making use of (2.1), we have

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)-f(x,y)|$$
$$\leq L_{n_1,n_2}(|f(t,s)-f(x_1,y_1)|;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$+|f(x_1,y_1)-f(x,y)|L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$\leq ML_{n_1,n_2}(|\frac{y_2}{1+y_2}-\frac{x_1}{1+x_1}|^{\alpha_1}|\frac{x_2}{1+x_2}-\frac{y_1}{1+y_1}|^{\alpha_2};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$+M|\frac{x}{1+x}-\frac{x_1}{1+x_1}|^{\alpha_1}|\frac{y}{1+y}-\frac{y_1}{1+y_1}|^{\alpha_2}L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y).$$

Now for $0\leq p\leq 1$, using $(a+b)^p\leq a^p+b^q$, one writes

$$|\frac{y_2}{1+y_2}-\frac{x_1}{1+x_1}|^{\alpha_1}\leq|\frac{y_2}{1+y_2}-\frac{x}{1+x}|^{\alpha_1}+|\frac{x}{1+x}-\frac{x_1}{1+x_1}|^{\alpha_1}$$

and

$$|\frac{x_2}{1+x_2}-\frac{y_1}{1+y_1}|^{\alpha_2}\leq|\frac{x_2}{1+x_2}-\frac{y}{1+y}|^{\alpha_2}+|\frac{y}{1+y}-\frac{y_1}{1+y_1}|^{\alpha_2}.$$

Using these inequalities in the above, we get

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)-f(x,y)|$$
$$\leq L_{n_1,n_2}(|\frac{y_2}{1+y_2}-\frac{x}{1+x}|^{\alpha_1}|\frac{x_2}{1+x_2}-\frac{y}{1+y}|^{\alpha_2}$$
$$;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)+|\frac{y}{1+y}-\frac{y_1}{1+y_1}|^{\alpha_2}L_{n_1,n_2}(|\frac{y_2}{1+y_2}$$
$$-\frac{x}{1+x}|^{\alpha_1};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$+|\frac{x}{1+x}-\frac{x_1}{1+x_1}|^{\alpha_1}L_{n_1,n_2}(|\frac{x_2}{1+x_2}-\frac{y_1}{1+y_1}|^{\alpha_2};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$

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$$+|\frac{x}{1+x}-\frac{x_1}{1+x_1}|^{\alpha_1}\times|\frac{y}{1+y}-\frac{y_1}{1+y_1}|^{\alpha_2}L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y).$$

Using the Hӧlder' inequality for $p_1=\frac{2}{\alpha_1},\quad p_2=\frac{2}{\alpha_2},\quad q_1=\frac{2}{2-\alpha_1},\quad q_2=\frac{2}{2-\alpha_2}$, we get

$L_{n_1,n_2}(|\frac{y_2}{1+y_2}-\frac{x}{1+x}|^{\alpha_1}|\frac{x_2}{1+x_2}-\frac{y}{1+y}|^{\alpha_2};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$

$=L_{n_1,n_2}(|\frac{y_2}{1+y_2}-\frac{x}{1+x}|^{\alpha_1};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)L_{n_1,n_2}(|\frac{x_2}{1+x_2}$

$-\frac{y}{1+y}|^{\alpha_2};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$

$\leq (L_{n_1,n_2}(\frac{y_2}{1+y_2}$

$-\frac{x}{1+x})^2\,;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)^{\frac{\alpha_1}{2}}\,(L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y))^{\frac{2-\alpha_1}{2}}(L_{n_1,n_2}(\frac{x_2}{1+x_2}$

$-\frac{y}{1+y})^2;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)^{\frac{\alpha_2}{2}}(L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y))^{\frac{2-\alpha_2}{2}}.$

This consequently gives the desired result. □

Remark 5.6 For $E=[0,\infty)$, we see that $d(x,E)=0$ and $d(y,E)=0$, so that we have

$$|L_{n_1,y_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)-f(x,y)| \leq M(p_{n_1}p_{n_2}q_{n_1}q_{n_2})^{4-\frac{\alpha_1+\alpha_2}{2}}\delta_{n_1}(x)^{\frac{\alpha_1}{2}}\delta_{n_2}(y)^{\frac{\alpha_2}{2}}.$$

Remark 5.7 By means of (2.1), it can be easily seen that $st-\lim_{n_1}\delta_{n_1}=0$ and $st-\lim_{n_2}\delta_{n_2}=0$. So, we can estimate the order of statistical approximation of our bivariate operators by means of Lipschitz type maximal functions using this result.

Also, as

$$\sup_{x\geq 0}\delta_{n_1}\leq\frac{p_1^{2n_1}q_1^{2n_1}}{[n_1^2+1]^2_{p_{n_1},q_{n_1}}}$$

and

$$p_1^{n_1}q_1^{n_1}(n_1+1)\leq(\frac{1}{p_1^{n_1}q_1^{n_1}}+\ldots+\frac{1}{p_{n_1}q_{n_1}}+1)p_1^{n_1}q_1^{n_1}$$

So, for $0\leq p_{n_1},q_{n_1}\leq 1$, we get

$$\frac{p_1^{2n_1}q_1^{2n_1}}{[n_1^2+1]^2_{p_{n_1},q_{n_1}}}\leq\frac{1}{[n_1^2+1]^2}.$$

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In a similar fashion we can obtain it for $\delta_{n_2}(y)$. So we have the following concluding remark.

Remark 5.8
(1) δ_{n_1} and δ_{n_2} approach to zero in statistical sense however they may not tend to zero in usual sense.
(2) In our case δ_{n_1} and δ_{n_2} approach to zero faster than that of classical BBH operators.

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A Study on Lateral Bases and Covered Lateral Ideals of po-bi Ternary Γ-Semigroups

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Abstract: In this paper, we define lateral bases and covered lateral ideals of po-bi ternary Γ semigroups. Also, we study some of their properties. Further, we explore the relationship between lateral base and covered lateral ideal of a po-bi ternary Γ semigroup.

Keywords: po-bi ternary $\Gamma-$ semigroup, Lateral base, Covered lateral ideal.
AMS Mathematics Subject Classification (2010): 20M12, 20N99.

1. Introduction and Preliminaries

Kasner [6] introduced the concept of n-ary algebras. In particular, n-ary semigroups are known as ternary semigroups for n=3 with one associative operation [7]. Ideal theory in ternary semigroup was studied by Sioson [8]. Quasi ideals and bi-ideals in ternary semigroups were studied by Dixit and Dewan [2].
Tamura [10] introduced the notion of (right) left base of semigroup. Later, Fabrici [3] gave the notion of one-sided bases of semigroups. Summaprad and Changphas [9] defined the relationship between right bases and maximal right ideals.
Fabrici [4] discussed some other properties and the mutual relation between covered ideals and bases of semigroups. The concept of one-sided bases of a ternary semigroup was given by Thongkam and Changphas [11]. Iampan [5] characterized the minimality and maximality of ordered laterals ideals in ternary semigroups. Akbar et al. [1] introduced po-bi ternary Γ-semigroup and studied the relationship between minimal generalized Γ-ideals and simple po-bi ternary Γ-semigroups.

To start with we need the following.
Definition 1.1 [1] A set T $(\neq \phi)$ is called a po-bi ternary $\Gamma-$ semigroup if it satisfies the following conditions:
(1). For any $x_1, x_2, x_3, x_4, x_5 \in T$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$ we have
$$[[x_1\gamma_1x_2\gamma_2x_3]\gamma_3x_4\gamma_4x_5] = [x_1\gamma_1[x_2\gamma_2x_3\gamma_3x_4]\gamma_4x_5] = [x_1\gamma_1x_2\gamma_2[x_3\gamma_3x_4\gamma_4x_5]],$$ and

(2). $\exists$ a po-bi relation $\leq$ so that
$a \leq b \Rightarrow a\gamma_1x\gamma_2y \leq b\gamma_1x\gamma_2y, x\gamma_1a\gamma_2y \leq x\gamma_1b\gamma_2y$ and $x\gamma_1y\gamma_2a \leq x\gamma_1y\gamma_2b$
for any $a, b, x, y \in T$ and $\gamma_1, \gamma_2 \in \Gamma$.

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Example 1.1 Let $T=\left\{\begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} : a,b,c \in N_0\right\}$. Here N_0, is the set of all non-negative integers and Γ be a non empty set such that $a\gamma b\gamma c = abc$ for all $\gamma \in \Gamma$ and $a,b,c \in T$.Then, under the regular multiplication of numbers with po-bi relation $\leq_{N_0}$ is "less than or equal to" T is a po-bi ternary Γ semigroup.

We now define po-bi relation $\leq_T$ on T, for any $\mathcal{A}$, $\mathcal{B} \in T$

$$\mathcal{A} \leq_T \mathcal{B} \text{ if and only if } a_{ij} \leq_{N_0} b_{ij}, \text{ for all } i \text{ and } j.$$

Then it is simple to verify that T is an po-bi ternary $\Gamma-$ semigroup under usual multiplication of matrices over N_0 with po-bi relation $\leq_T$.

For $H \subseteq T$ we denote $(H]$ the subset of T defined by

$$(H] = \{s \in T | s \leq h, \text{ for some } h \in H\}$$

Theorem 1.1 [1] Let T be a po-bi ternary $\Gamma-$ semigroup, then the following conditions hold

(1) $\mathcal{A} \subseteq (\mathcal{A}]$, for all $\mathcal{A} \subseteq T$.
(2) If $\mathcal{A} \subseteq \mathcal{B} \subseteq S$, then $(\mathcal{A}] \subseteq (\mathcal{A}]$.
(3) $((\mathcal{A}]] = (\mathcal{A}]$, for all $\mathcal{A} \subseteq T$.
(4) $(\mathcal{A}]\Gamma(\mathcal{B}]\Gamma(\mathcal{C}] \subseteq (\mathcal{A}\Gamma\mathcal{B}\Gamma\mathcal{C}]$, for all $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C} \subseteq T$.

Definition 1.2 [1] If T is a po-bi ternary $\Gamma-$ semigroup and S is a ternary $\Gamma-$ subsemigroup of T, then S is a po-bi ternary sub $\Gamma-$ semigroup of T if $(S] \subseteq S$.

Definition 1.3 [1] If T is a po-bi ternary $\Gamma-$ semigroup and M be a non-empty subset of T then M is called a po-bi lateral ideal of T if $T\Gamma M\Gamma T \subseteq M$ and $(M] \subseteq M$.

Definition 1.4 A po-bi ternary $\Gamma-$ semigroup T is called lateral simple if it has no proper lateral ideals.

Definition 1.5 A po-bi lateral ideal M of T is called a minimal po-bi lateral $\Gamma-$ ideal of T if it does not contain any lateral $\Gamma-$ ideal of T.

Definition 1.6 A proper po-bi lateral ideal M of a po-bi ternary $\Gamma-$ semigroup T is called a maximal po-bi lateral ideal of T if for any po-bi lateral ideal $\mathcal{A}$ of T such that $M \subset \mathcal{A}$, then $\mathcal{A} = T$. Analogously, if for any proper po-bi lateral ideal $\mathcal{A}$ of T such that $M \subseteq \mathcal{A}$, then $\mathcal{A} = \mathcal{M}$.

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Lemma 1.1 For any subset $\mathcal{A}(\neq \phi)$ of T, $(T\Gamma T\Gamma\mathcal{A}\Gamma T\Gamma T \cup T\Gamma\mathcal{A}\Gamma T \cup \mathcal{A}]$ is the smallest lateral ideal of T containing $\mathcal{A}$.

Also, for any $a \in T$, $M(a) = (T\Gamma T\Gamma a\Gamma T\Gamma T \cup T\Gamma a\Gamma T \cup a]$.
Let T be a po-bi ternary $\Gamma-$ semigroup and $a,\ b\ (\neq 0)$ be any elements of T. Then the equivalence relation on M is defined by:

$$a \text{ M } b \text{ if } a \text{ and } b \text{ generate the same principal lateral ideal of } T.$$

M-class containing a is denoted by $\mathcal{M}_a$.

Now let $a,\ b\ (\neq 0)$ be any elements of T, then we define a quasi-ordering $\preceq$ on the set of all equivalence classes as:

$$\mathcal{M}_a \preceq \mathcal{M}_b \text{ if and only if } \mathcal{M}(a) \subseteq \mathcal{M}(b),$$

where $\mathcal{M}(a)$ and $\mathcal{M}(b)$ are the principal lateral ideals of T generated by a and b respectively.

2. Lateral Bases of po-bi Ternary $\Gamma-$ Semigroups

Definition 2.1 Let T be a po-bi ternary $\Gamma-$ semigroup. A subset $\mathcal{A}$ of T is called a lateral base of T if,
(1) $(\mathcal{A} \cup T\Gamma\mathcal{A}\Gamma T \cup T\Gamma T\Gamma\mathcal{A}\Gamma T\Gamma T] = T$, and
(2) $(\mathcal{B} \cup T\Gamma\mathcal{B}\Gamma T \cup T\Gamma T\Gamma\mathcal{B}\Gamma T\Gamma T] \neq T$, for any proper subset $\mathcal{B}$ of $\mathcal{A}$.

Example 2.1 Let $T=\left\{\begin{pmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{pmatrix} : a,b,c,d \in N_0\right\}$ and $\Gamma\ (\neq \phi)$ be a set such that $a\gamma b\gamma c = abc$ for all $\gamma \in \Gamma$ and $a,b,c \in T$. Then T is a po-bi ternary $\Gamma-$ semigroup under the same operation as defined in the Example 1.1.
Now consider a set $\mathcal{M} = \left\{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$. Then we can easily verify that $\mathcal{M}$ is lateral base of T as $(\mathcal{M} \cup T\Gamma\mathcal{M}\Gamma T \cup T\Gamma T\Gamma\mathcal{M}\Gamma T\Gamma T] = T$.

Lemma 2.1 If two elements $a,\ b$ belongs to lateral base $\mathcal{A}$ of a po-bi ternary $\Gamma-$ semigroup T such that $a \in (T\Gamma b\Gamma T \cup T\Gamma T\Gamma b\Gamma b\Gamma T\Gamma T]$, then $a = b$.

Proof. Let $\mathcal{A}$ be a lateral base of T and $a,\ b \in \mathcal{A}$ such that $a \in (T\Gamma b\Gamma T \cup T\Gamma T\Gamma b\Gamma T\Gamma T]$. Suppose $a \neq b$. Take a proper subset $\mathcal{B}$ of $\mathcal{A}$ as $\mathcal{B} = \mathcal{A} \setminus \{a\}$. Then, b

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belongs to $\mathcal{B}$. Now $a \in (T\Gamma b\Gamma T \cup T\Gamma T\Gamma b\Gamma T\Gamma T]$, it implies

$$\begin{aligned} \mathcal{M}(a) &\subseteq (T\Gamma b\Gamma T \cup T\Gamma T\Gamma b\Gamma T\Gamma T] \\ &\subseteq \mathcal{M}(b) \\ &\subseteq \mathcal{M}(B). \end{aligned}$$

Hence $\mathcal{M}(A) \subseteq \mathcal{M}(B)$. Thus, we get $\mathcal{M}(B) = T$, which is contradiction because $\mathcal{A}$ be a lateral base of T. Therefore $a = b$.

Lemma 2.2 Let $\mathcal{A}$ be a subset of a po-bi ternary $\Gamma-$ semigroup T. Then $\mathcal{A}$ is a lateral base of T iff the following conditions hold:
(i) for any $t \in T$, $\exists$ a in $\mathcal{A}$ such that $\mathcal{M}_t \preceq \mathcal{M}_a$;
(ii) If $a_1, a_2 \in \mathcal{A}$ such that $a_1 \neq a_2$, then neither $\mathcal{M}_{a_1} \preceq \mathcal{M}_{a_2}$ nor $\mathcal{M}_{a_2} \preceq \mathcal{M}_{a_1}$

Proof. Suppose $\mathcal{A}$ is a lateral base of T and Let $t \in T$. Then obviously $(\mathcal{A} \cup T\Gamma\mathcal{A}\Gamma T \cup T\Gamma T\Gamma\mathcal{A}\Gamma T\Gamma T] = T$ and $t \in (\mathcal{A} \cup T\Gamma\mathcal{A}\Gamma T \cup T\Gamma T\Gamma\mathcal{A}\Gamma T\Gamma T]$, so $t \in (\mathcal{A}]$ or $t \in (T\Gamma\mathcal{A}\Gamma T]$ or $t \in (T\Gamma T\Gamma\mathcal{A}\Gamma T\Gamma T]$. If $t \in (\mathcal{A}]$, then $t \leq a$, for some $a \in \mathcal{A}$, it follows $\mathcal{M}_t \preceq \mathcal{M}_a$. If $t \in (T\Gamma\mathcal{A}\Gamma T]$, then $t \leq t_1\gamma_1 a'\gamma_2 t_2$ for some $t_1, t_2 \in T$,$\gamma_1, \gamma_2 \in \Gamma$ and $a' \in \mathcal{A}$, we have $\mathcal{M}_t \preceq \mathcal{M}_{a'}$. If $t \in (T\Gamma T\mathcal{A}\Gamma T\Gamma T]$, then $t \leq t_3\gamma_3 t_4\gamma_4 a''\gamma_5 t_5\gamma_6 t_6$ for some $t_3, t_4, t_5, t_6 \in T$,$\gamma_3, \gamma_4, \gamma_5, \gamma_6 \in \Gamma$ and $a'' \in \mathcal{A}$, we obtain $\mathcal{M}_t \preceq \mathcal{M}_{a''}$. Hence the result (i) is proved. Now suppose $a_1 \leq a_2$ i.e. $\mathcal{M}_{a_1} \preceq \mathcal{M}_{a_2}$, $\mathcal{M}(a_1) \subseteq \mathcal{M}(a_2)$. If $a_1 \neq a_2$, then $a_1 \in (T\Gamma a_2\Gamma T \cup T\Gamma T\Gamma a_2\Gamma T\Gamma T]$. So, by Lemma 2.1, $a_1 = a_2$. Thus for $a_1 \neq a_2$, $\mathcal{M}_{a_1} \npreceq \mathcal{M}_{a_2}$. Analogously, we can prove $\mathcal{M}_{a_2} \npreceq \mathcal{M}_{a_1}$.

Conversely assuming that conditions (i) and (ii) holds. From (i), $\mathcal{M}(\mathcal{A}) = T$. To show that $\mathcal{A}$ is lateral base of T, it is sufficient to show that $\exists$ no proper subsets $\mathcal{B}$ of $\mathcal{A}$ s.t.$\mathcal{M}(\mathcal{B}) = T$, where $\mathcal{B}$ is a proper subset of $\mathcal{A}$. Let $a_1 \in \mathcal{A} \setminus \mathcal{B}$, then $\exists$ $a_2 \in \mathcal{B}$ such that $a_1 \in (a_2 \cup T\Gamma a_2\Gamma T \cup T\Gamma T\Gamma a_2\Gamma T\Gamma T]$. Hence $\mathcal{M}(a_1) \subseteq \mathcal{M}(a_2)$, i.e. $\mathcal{M}_{a_1} \preceq \mathcal{M}_{a_2}$, which is a contradiction to the assumption (2). Therefore $\mathcal{A}$ is a lateral base of T.

Theorem 2.1 Let a po-bi ternary $\Gamma-$ semigroup T contains a lateral ideal. Then $\mathcal{M}$ is a maximal lateral ideal of T iff $T \setminus \mathcal{M}$ is a maximal M-class.

Proof. Suppose $\mathcal{M}$ is a maximal lateral ideal of T. Let $t_1, t_2 \in T \setminus \mathcal{M}$. As $\mathcal{M} \subseteq \mathcal{M} \cup \mathcal{M}(t_1) \subseteq T$, it implies $\mathcal{M} \cup \mathcal{M}(t_1) = T$. thus $t_2 \in \mathcal{M}(t_1)$. Similarly, $t_1 \in \mathcal{M}(t_2)$. Therefore $\mathcal{M}(t_1) = \mathcal{M}(t_2)$. This shows that $T \setminus \mathcal{M}$ is in M-class. Now for $T \setminus \mathcal{M} \prec \mathcal{M}_t$ for some $t \in T$, then $\exists$ $s \in T$ such that $T \setminus \mathcal{M} \subseteq \mathcal{M}(s)$ $\mathcal{M}$, which is a contradiction. Hence $T \setminus \mathcal{M}$ is a maximal M-class.

Conversely assume that $T \setminus \mathcal{M}$ is a maximal **M**-class such that $T \setminus \mathcal{M} = \mathcal{M}_t$ for some $t \in T$. We need to show that $\mathcal{M}$ is maximal lateral ideal of T. Firstly, we show that $\mathcal{M}$ is a lateral ideal of T. i.e. $T\Gamma\mathcal{M}\Gamma T \cup T\Gamma T\Gamma\mathcal{M}\Gamma T\Gamma T \subseteq \mathcal{M}$ and $(\mathcal{M}] = \mathcal{M}$. On contrary

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suppose that $\mathcal{M}$ is not a lateral ideal of T. Then, $\exists\ m \in T\Gamma\mathcal{M}\Gamma T \cup T\Gamma T\Gamma\mathcal{M}\Gamma T\Gamma T$ such that $m \notin \mathcal{M}$. It implies $m \in T \setminus \mathcal{M} = \mathcal{M}_t$, then we have $\mathcal{M}(m) = \mathcal{M}(t)$. As $m \in T\Gamma\mathcal{M}\Gamma T \cup T\Gamma T\Gamma\mathcal{M}\Gamma T\Gamma T$, there exists $t_1, t_2, ..., t_6 \in T$,$\gamma_1, \gamma_2, ..., \gamma_6 \in \Gamma$ and $m_1, m_2 \in \mathcal{M}$ such that $m = t_1\gamma_1 m_1\gamma_2 t_2$ or $m = t_3\gamma_3 t_4\gamma_4 m_2\gamma_5 t_5\gamma_6 t_6$. Thus, we have $\mathcal{M}_t \prec \mathcal{M}_{m_1}$ or $\mathcal{M}_t \prec \mathcal{M}_{m_2}$. Therefore, $\mathcal{M}_t \prec \mathcal{M}_m$, which is a countering to the assumption that $T \setminus \mathcal{M}$ is a maximal **M**-class. It implies $T\Gamma\mathcal{M}\Gamma T \cup T\Gamma T\Gamma\mathcal{M}\Gamma T\Gamma T \subseteq \mathcal{M}$. Now, let $m \in \mathcal{M}$ and $s \in T$ so as $s \leq m$. Suppose $s \in T \setminus \mathcal{M} = \mathcal{M}_t$. Then $s < m$, threfore $\mathcal{M}_t \prec \mathcal{M}_m$. This is a contradiction. Thus, we get $s \in \mathcal{M}$. Consequently, $\mathcal{M}$ is a lateral ideal of T. Let $\mathcal{M}'$ be a proper lateral ideal of T so that $\mathcal{M}$ is properly contained in $\mathcal{M}'$. Then $\exists\ a \in T \setminus \mathcal{M}'$. Thus, $\mathcal{M}_t = \mathcal{M}_a$. Also there exists $m' \in \mathcal{M}' \setminus \mathcal{M}$ so that $\mathcal{M}_{m'} = \mathcal{M}_t$. It follows that $a \in \mathcal{M}_a = \mathcal{M}_t = \mathcal{M}_{m'} \subseteq \mathcal{M}$, which is also not possible. Hence $\mathcal{M}$ is a maximal lateral ideal of T.

3. Covered Lateral Ideals of po-bi Ternary $\Gamma-$ Semigroups

Definition 3.1 A proper lateral ideal $\mathcal{L}$ of T is known as a covered lateral ideal of (CLt. ideal) if $\mathcal{L} \subset (T\Gamma(T-\mathcal{L})\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{L})\Gamma T\Gamma T]$.

Remark 3.1 By definition of CLt.-ideal, po-bi Ternary $\Gamma-$ ternary semigroup itself is not a CLt.-ideal.

Example 3.1 Let $T = \left\{\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} : a, b, c, d, e, f \in N_0\right\}$ excluding null matrix, with po-bi relation $\leq_{N_0}$ is "less than or equal to" and Γ be a non empty set such that $a\gamma b\gamma c = abc$ for all $\gamma \in \Gamma$ and $a, b, c \in T$. Then T is a po-bi ternary Γ semigroup under the same operation as defined in the Example 1.1.

Consider $\mathcal{L} = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} : b \in N\right\}$. Then it easy to verify that $\mathcal{L}$ is lateral ideal of T and $T - \mathcal{L} = \left\{\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & d & e \end{pmatrix} : a, b, c, d, e \in N_0\right\}$.

Now $\mathcal{L} = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} : b \in N\right\} \subset \left\{\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} : a, b, c, d, e, f \in N_0\right\} =$ $(T\Gamma(T-\mathcal{L})\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{L})\Gamma T\Gamma T]$. Therefore $\mathcal{L}$ is a CLt.-ideal of T.

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Lemma 3.1 If a po-bi ternary $\Gamma -$ semigroup T have two different lateral ideals $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1 \cup \mathcal{L}_2 = T$, then none of the lateral ideals $\mathcal{L}_1$, $\mathcal{L}_2$ is a CLt. -ideal.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lateral ideals a po-bi ternary $\Gamma -$ semigroup T such that $\mathcal{L}_1 \cup \mathcal{L}_2 = T$, it implies T - $\mathcal{L}_1 \subset \mathcal{L}_2$ and T - $\mathcal{L}_2 \subset \mathcal{L}_1$. Suppose that $\mathcal{L}_2$ is a covered lateral ideal of T, then $\mathcal{L}_2 \subset (T\Gamma(T-\mathcal{L}_2)\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{L}_2)\Gamma T\Gamma T]$ which implies $\mathcal{L}_2 \subset (T\Gamma(T-\mathcal{L}_2)\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{L}_2)\Gamma T\Gamma T] \subset (T\mathcal{L}_1\Gamma T \cup T\Gamma T\mathcal{L}_1\Gamma T\Gamma T] \subset (T\mathcal{L}_1\Gamma T \cup T\mathcal{L}_1\Gamma T] \subseteq \mathcal{L}_1$. Similarly, we can show that $\mathcal{L}_1 \subset \mathcal{L}_2$. Therefore $\mathcal{L}_1 = \mathcal{L}_2$. But $\mathcal{L}_1$ and $\mathcal{L}_2$ are different, which is a contradiction. As a result, none of $\mathcal{L}_1$ or $\mathcal{L}_2$ is a CLt. -ideal.

Corollary 3.1 If a po-bi ternary $\Gamma -$ semigroup T contains more than one maximal lateral ideal, then none of the maximal lateral ideal is a CLt.-ideal.

Proof. Assume that T have two maximal lateral ideals $\mathcal{M}_1$ and $\mathcal{M}_2$. Since the union of any two lateral ideals is a lateral ideal. So as $\mathcal{M}_1 \cup \mathcal{M}_2$ is a lateral ideal of T and $\mathcal{M}_1 \subset \mathcal{M}_1 \cup \mathcal{M}_2$ and $\mathcal{M}_1$ is a maximal lateral ideal of T. Therefore $\mathcal{M}_1 \cup \mathcal{M}_2 = T$. Hence by Lemma 3.1, neither $\mathcal{M}_1$ nor $\mathcal{M}_2$ is a CLt.-ideal of T

Lemma 3.2 If $\mathcal{L}$ is a lateral ideal of T suc that $\mathcal{L} \subset (T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T]$ and $\mathcal{L} \neq (T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T]$ for some $t \in T$. Consequently, $\mathcal{L}$ is a CLt. ideal of T.

Proof. Assume that $\mathcal{L}$ is a lateral ideal of T such that $\mathcal{L} \subset (T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T]$ and $\mathcal{L} \neq (T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T]$ for some $t \in T$. Then $t \notin \mathcal{L}$, else $(T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T] \subseteq (T\Gamma\mathcal{L}\Gamma T \cup T\Gamma T\Gamma\mathcal{L}\Gamma T\Gamma T] \subseteq \mathcal{L}$, therefore $\mathcal{L} \neq (T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T]$. Thus $\mathcal{L} \subset (T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T] \subset (T\Gamma(T-\mathcal{L})\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{L})\Gamma T\Gamma T]$. Hence $\mathcal{L}$ is a CLt. ideal of T.

Corollary 3.2 A po-bi ternary $\Gamma -$ semigroup of T in which t does not belongs to $(T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T]$ has a CLt.-ideal.

Proof. Let $\mathcal{L} = (T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T]$. Indeed $\mathcal{L}$ is a lateral ideal of T. If $t \notin \mathcal{L}$, we get $\mathcal{L} = (T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T] \subset (T\Gamma(T-\mathcal{L})\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{L})\Gamma T\Gamma T]$. It means $\mathcal{L}$ is a CLt.-ideal of T.

Lemma 3.3 If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two covered lateral ideal of a po-bi ternary $\Gamma -$ semigroup T. Then $\mathcal{L}_1 \cup \mathcal{L}_2$ is a CLt.-ideal of T.

Proof. To prove $\mathcal{L}_1 \cup \mathcal{L}_2$ is a CLt.-ideal of T. We need to show that $\mathcal{L}_1 \cup \mathcal{L}_2 \subset (T\Gamma[T-(\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T \cup T\Gamma T\Gamma[T-(\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T\Gamma T]$. As CLt.-ideal is $\mathcal{L}_1$ i.e. $\mathcal{L}_1 \subset (T\Gamma(T-\mathcal{L}_1)\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{L}_1)\Gamma T\Gamma T]$, which implies for any $m \in \mathcal{L}_1$, there exists $m_1, m_2 \in T-\mathcal{L}_1$ such that $m \in (T\Gamma m_1\Gamma T \cup T\Gamma T\Gamma m_2\Gamma T\Gamma T]$. Now we have following four cases:

Case 1: If $m_1, m_2 \in T - \mathcal{L}_1 - \mathcal{L}_2$. Then $m \in (T\Gamma(T - \mathcal{L}_1 - \mathcal{L}_2)\Gamma T \cup T\Gamma T\Gamma(T - \mathcal{L}_1 - \mathcal{L}_2)\Gamma T\Gamma T] \subseteq (T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T \cup T\Gamma T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T\Gamma T]$.

Case 2: If $m_1, m_2 \in (T - \mathcal{L}_1) \cap \mathcal{L}_2$. It implies $m_1, m_2 \in \mathcal{L}_2 \subset (T\Gamma(T - \mathcal{L}_2)\Gamma T \cup T\Gamma T\Gamma(T - \mathcal{L}_2)\Gamma T\Gamma T]$. Then there exists $m_3, m_4, m_5, m_6 \in T - \mathcal{L}_2$ s.t. $m_1 \in (T\Gamma m_3\Gamma T \cup T\Gamma T\Gamma m_4\Gamma T\Gamma T]$ and $m_2 \in (T\Gamma m_5\Gamma T \cup T\Gamma T\Gamma m_6\Gamma T\Gamma T]$. Here $m_3, m_4 \notin \mathcal{L}_1$, otherwise $m_1 \in (T\Gamma m_3\Gamma T \cup T\Gamma T\Gamma m_4\Gamma T\Gamma T] \subseteq (T\mathcal{L}_1\Gamma T \cup T\Gamma T\mathcal{L}_1\Gamma T\Gamma T] \subseteq \mathcal{L}_1$. Therefore $m_1 \in \mathcal{L}_1$, which is contradiction as $m_1 \in T - \mathcal{L}_1$. Thus, we have $m_3, m_4 \in T - \mathcal{L}_1$. Therefore $m_3, m_4 \in T - \mathcal{L}_1 \cap T - \mathcal{L}_2 = T - (\mathcal{L}_1 \cup \mathcal{L}_2)$. Similarly, $m_5, m_6 \in T - (\mathcal{L}_1 \cup \mathcal{L}_2)$. Now

$$
\begin{aligned}
m &\in (T\Gamma m_1\Gamma T \cup T\Gamma T\Gamma m_2\Gamma T\Gamma T] \\
&\subset (T\Gamma(T\Gamma m_3\Gamma T \cup T\Gamma T\Gamma m_4 T\Gamma T]\Gamma T \cup T\Gamma T\Gamma(T\Gamma m_5\Gamma T \cup T\Gamma T\Gamma m_6\Gamma T\Gamma T]\Gamma T\Gamma T] \\
&\subseteq ((T]\Gamma(T\Gamma m_3\Gamma T \cup T\Gamma T\Gamma m_4\Gamma T\Gamma T]\Gamma(T] \cup (T]\Gamma(T]\Gamma(T\Gamma m_5\Gamma T \cup T\Gamma T\Gamma m_6\Gamma T\Gamma T]\Gamma(T]\Gamma(T]] \\
&\subseteq ((T\Gamma(T\Gamma m_3\Gamma T \cup T\Gamma T\Gamma m_4\Gamma T\Gamma T)\Gamma T \cup T\Gamma T\Gamma(T\Gamma m_5\Gamma T \cup T\Gamma T\Gamma m_6\Gamma T\Gamma T)\Gamma T\Gamma T]] \\
&= (T\Gamma(T\Gamma m_3\Gamma T \cup T\Gamma T\Gamma m_4\Gamma T\Gamma T)\Gamma T \cup T\Gamma T\Gamma(T\Gamma m_5\Gamma T \cup T\Gamma T\Gamma m_6\Gamma T\Gamma T)\Gamma T\Gamma T] \\
&\subseteq (T\Gamma T\Gamma m_3\Gamma T\Gamma T \cup T\Gamma m_4\Gamma T \cup T\Gamma m_5\Gamma T \cup T\Gamma T\Gamma m_6\Gamma T\Gamma T] \\
&\subset (T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T \cup T\Gamma T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T\Gamma T].
\end{aligned}
$$

Case 3: If $m_1 \in T - \mathcal{L}_1 - \mathcal{L}_2$ and $m_2 \in (T - \mathcal{L}_1) \cap \mathcal{L}_2$. It implies $m_1 \in T - (\mathcal{L}_1 \cup \mathcal{L}_2)$. From case 2, $m_2 \in T - (\mathcal{L}_1 \cup \mathcal{L}_2)$. Therefore $m \in (T\Gamma m_1\Gamma T \cup T\Gamma T\Gamma m_2\Gamma T\Gamma T] \subset (T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T \cup T\Gamma T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T\Gamma T]$.

Case 4: If $m_2 \in T - \mathcal{L}_1 - \mathcal{L}_2$ and $m_1 \in (T - \mathcal{L}_1) \cap \mathcal{L}_2$. This is then the same case 3. Consequently, in all these cases $m \in (T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T \cup T\Gamma T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T\Gamma T]$. Likewise, we can prove this for $m \in \mathcal{L}_2$. Thus $\mathcal{L}_1 \cup \mathcal{L}_2 \subset (T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T \cup T\Gamma T\Gamma[T - (\mathcal{L}_1 \cup \mathcal{L}_2)]\Gamma T\Gamma T]$ and hence $\mathcal{L}_1 \cup \mathcal{L}_2$ is a CLt.-ideal of T.

Lemma 3.4 If $\mathcal{L}_1$ is a covered lateral ideal of a po-bi ternary $\Gamma -$ semigroup T and $\mathcal{L}_2$ is any lateral ideal of T. Then $\mathcal{L}_1 \cap \mathcal{L}_2$ is a CLt.-ideal of T, provided that $\mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset$.

Proof. Suppose this $\mathcal{L}_1$ is a covered lateral ideal and $\mathcal{L}_2$ is a lateral ideal of T such that $\mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset$. Then $\mathcal{L}_1 \subset (T\Gamma(T - \mathcal{L}_1)\Gamma T \cup T\Gamma T\Gamma(T - \mathcal{L}_1)\Gamma T\Gamma T]$. It implies

$$
\begin{aligned}
\mathcal{L}_1 \cap \mathcal{L}_2 &\subset (T\Gamma(T - \mathcal{L}_1)\Gamma T \cup T\Gamma T\Gamma(T - \mathcal{L}_1)\Gamma T\Gamma T] \\
&\subset (T\Gamma[T - (\mathcal{L}_1 \cap \mathcal{L}_2)]\Gamma T \cup T\Gamma T\Gamma[T - (\mathcal{L}_1 \cap \mathcal{L}_2)]\Gamma T\Gamma T].
\end{aligned}
$$

Therefore, $\mathcal{L}_1 \cap \mathcal{L}_2$ is a CLt.-ideal of T.

Lemma 3.5 If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two covered lateral ideal of a po-bi ternary $\Gamma -$ semigroup T. Then $\mathcal{L}_1 \cap \mathcal{L}_2$ is a CLt.-ideal of T, provided that $\mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset$.

Proof. Proof is similar to the lemma 3.4.

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Definition 3.2 A proper lateral ideal $\mathcal{L}$ is called the greatest lateral ideal of a po-bi ternary $\Gamma-$ semigroup T if $\mathcal{L}$ contains any proper lateral ideal of T, to be denoted by $\mathcal{L}^*$.

Theorem 3.1 If a po-bi ternary $\Gamma-$ semigroup T have only one maximal lateral ideal $\mathcal{M}$ and $\mathcal{M}$ is a CLt.-ideal, then $\mathcal{M} = \mathcal{M}^*$

Proof. Suppose that $\mathcal{M}$ is a maximal lateral of T and $\mathcal{M}$ is also a CLt.-ideal of T. Let $\mathcal{M}_1$ is a lateral ideal of T, if possible $\mathcal{M}_1 \nsubseteq \mathcal{M}$. As $\mathcal{M}_1 \cup \mathcal{M}$ is a lateral ideal of T and that is $\mathcal{M} \subset \mathcal{M}_1 \cup \mathcal{M}$. It follows that $\mathcal{M}_1 \cup \mathcal{M} = T$. Then by Lemma 3.1, T cannot contains any CLt.-ideals, which is contradiction. Thus $\mathcal{M}_1 \subseteq \mathcal{M}$. Therefore $\mathcal{M} = \mathcal{M}^*$.

Theorem 3.2 If T is not a simple po-bi ternary $\Gamma-$ semigroup such that it does not contain any two proper lateral ideals in which there is an empty intersection. Then, T contains at least one CLt.-ideal.

Proof. Let $\mathcal{M}$ be a proper lateral ideal of T. Then, $\mathcal{M}_1 = (T\Gamma(T-\mathcal{M})\Gamma T \cup T\Gamma T(T-\mathcal{M})\Gamma T\Gamma T]$ is also a lateral ideal of T. By hypothesis $\mathcal{M} \cap \mathcal{M}_1 \neq \emptyset$. Thus $\mathcal{M}_c = \mathcal{M} \cap \mathcal{M}_1$ is a lateral ideal of T and $\mathcal{M}_c \subset \mathcal{M}$, it implies $T-\mathcal{M}_c \supset T-\mathcal{M}$.

Now, $\mathcal{M}_c \subset \mathcal{M}_1 = (T\Gamma(T-\mathcal{M})\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{M})\Gamma T\Gamma T] \subset (T\Gamma(T-\mathcal{M}_c)\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{M}_c)\Gamma T\Gamma T]$. It implies that $\mathcal{M}_c$ is a CLt. ideal of T.

Theorem 3.3 Let T be a po-bi ternary $\Gamma-$ semigroup containing maximal lateral ideals. If the intersection of maximal lateral ideal is either empty or a covered lateral ideal, then T contains a lateral base.

Proof. Let $\{\mathcal{M}_i : i \in I\}$ be the set of all maximal lateral ideals of T. By the Theorem 2.1, for each $i \in I$, $T - \mathcal{M}_i$ is a maximal **M**-class. Set $T - \mathcal{M}_i = \mathcal{M}_{m_i}$, for each $i \in I$.
Then $\mathcal{M}_{int} = \bigcap_{i\in I} \mathcal{M}_i = \bigcap_{i\in I} (T-\mathcal{M}_{m_i}) = T - \bigcup_{i\in I} \mathcal{M}_{m_i}$.
Construct $\mathcal{C}$ as, for every $\mathcal{M}_{m_i}$, put into $\mathcal{C}$ only one element in $\mathcal{C}$. We prove that $\mathcal{C}$ is a lateral base of T. Now we consider two cases:

$Case 1$: $\mathcal{M}_{int} = \emptyset$. Then $T = \bigcup_{i\in I} \mathcal{M}_{m_i}$. If $m \in T$, then $m \in \mathcal{M}_{m_i}$ for some $i \in I$ and so $\mathcal{M}\Gamma(m) = \mathcal{M}\Gamma(m_i)$. Then $\mathcal{M}_m \preceq \mathcal{M}_{m_i}$. As $\mathcal{M}_{m_i}$ is a maximal **M**-class for all $i \in I$, it implies for different $i, j \in I$, neither $\mathcal{M}_{m_i} \preceq \mathcal{M}_{m_j}$ nor $\mathcal{M}_{m_j} \preceq \mathcal{M}_{m_i}$. By Lemma 2.2, $\mathcal{C}$ is a lateral base of T.

$Case 2$: If $\mathcal{M}_{int}$ is a covered lateral ideal of T, i.e. $\mathcal{M}_{int} \subseteq (T\Gamma(T-\mathcal{M}_{int})\Gamma T \cup T\Gamma T\Gamma(T-\mathcal{M}_{int})\Gamma T\Gamma T]$. If $m \in T - \mathcal{M}_{int}$, then $m \in \bigcup_{i\in I} \mathcal{M}_{m_i}$ and so $m \in \mathcal{M}_{m_{i_0}}$ for

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some $i_0 \in I$. Now $\mathcal{M}\Gamma(m) = \mathcal{M}\Gamma(m_{i_0}) \subseteq (\mathcal{C} \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\Gamma\mathcal{C}\Gamma T\Gamma T]$. Thus, we have $m \in (\mathcal{C} \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\Gamma\mathcal{C}\Gamma T\Gamma T]$. It implies $T - \mathcal{M}_{int} \subseteq (\mathcal{C} \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\Gamma\mathcal{C}\Gamma T\Gamma T]$.
Also

$$\begin{aligned}
\mathcal{M}_{int} &\subseteq (T\Gamma(T - \mathcal{M}_{int})\Gamma T \cup T\Gamma T\Gamma(T - \mathcal{M}_{int})T\Gamma T] \\
&\subseteq (T\Gamma(\mathcal{C} \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\Gamma\mathcal{C}\Gamma T\Gamma T]\Gamma T \cup T\Gamma T(\mathcal{C} \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\mathcal{C}\Gamma T\Gamma T])\Gamma T\Gamma T] \\
&\subseteq ((T]\Gamma(\mathcal{C} \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\Gamma\mathcal{C}\Gamma T\Gamma T]\Gamma(T] \cup (T]\Gamma(T]\Gamma(\mathcal{C} \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\Gamma\mathcal{C}\Gamma T\Gamma T]\Gamma(T]\Gamma(T]] \\
&\subseteq ((T\Gamma(\mathcal{C} \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\Gamma\mathcal{C}\Gamma T\Gamma T)\Gamma T \cup T\Gamma T(\mathcal{C}\Gamma \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\mathcal{C}\Gamma T\Gamma T)\Gamma T\Gamma T]] \\
&= (T\Gamma(\mathcal{C} \cup T\mathcal{C}\Gamma T \cup T\Gamma T\mathcal{C}\Gamma T\Gamma T)\Gamma T \cup T\Gamma T\Gamma(\mathcal{C} \cup T\mathcal{C}\Gamma T \cup T\Gamma T\mathcal{C}\Gamma T\Gamma T)\Gamma T\Gamma T] \\
&\subseteq (T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\mathcal{C}\Gamma T\Gamma T].
\end{aligned}$$

It follows that $T = \mathcal{M}_{int} \cup (T - \mathcal{M}_{int}) \subseteq (\mathcal{C} \cup T\Gamma\mathcal{C}\Gamma T \cup T\Gamma T\Gamma\mathcal{C}\Gamma T\Gamma T]$.
This implies if $m \in T$, then there exists $m_i \in \mathcal{C}$ such that $\mathcal{M}_m \preceq \mathcal{M}_{m_i}$. Hence by Lemma 2.2 $\mathcal{C}$ is a lateral base of T.

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Study on three unequal straight cracks with coalesced yield zones: A case of parabolic stress distribution

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Abstract
Load carrying capacity of an infinite isotropic elastic perfectly plastic plate has been investigated in this paper when three unequal straight hairline cracks with unified yield zones weakens the plate. Effect of quadratically varying mechanical loading on the yield zones have been studied. Well-known complex variable method is applied to find the solution to the problem. Analytical expressions are derived for stress intensity factors and applied stresses. These expressions are compared with the existing solution of two equal straight cracks, which is to be considered as a limiting case of the problem discussed in this paper.

Keywords: Dugdale model; Yield zone; Stress intensity factor; Multiple crack;
[2010] MSC: code 74R05, 74R10

1. Introduction
The presence of cracks is a threat to any structure due to its catastrophic nature. Structure containing cracks may survive, but the strength of the structure will decrease [1] in the presence of these cracks. Therefore, it must be important to study the behaviour of load bearing capacity of structures having such defects. Moreover, it was pointed out by Gdoutos [2] that occasionally the structures may fail at the stress level that is below the material yield stress. Therefore, the present paper in an attempt to analyze the load-carrying capacity of an isotropic infinite plate weakened by multiple cracks with unified yield zones under quadratically varying mechanical stress distribution. The reason behind taking quadratically varying stress distribution is to model mathematically such a situation where stress assumed on the yield zones was below the material yield stress, as mention above.

There is a long history behind the calculation of load-carrying capacity of the structures when crack or crack like defects exist. In his famous paper Dugdale [3] modelled development of plastic zones at the crack tips under general yielding conditions. Since the model was mathematically simple, therefore, various researchers used and modified the model for different kinds of materials [4] and mechanical loading conditions [5], [6] and many others. Dugdale model not been limited to a single crack problem but widely applied to study the problem of multiple cracks. Theocaris [7] used Dugdale hypothesis to study two equal/ unequal symmetrically situated straight cracks under normal yielding conditions and supplies

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closed form expression for applied load ratio using complex variable method. Furthermore, Bhargava et al. [8] studied two unequal asymmetric collinear straight cracks under parabolic stress distribution for sperate yield zones and for coalesced yield zones in [9] using Dugdale hypothesis. Strip yield model was discussed by Collins et al. [10] for two equal straight cracks in an infinite plate and for unequal cracks by Nishimura [11]. Complex variable technique is successfully used to find closed form solution for multiple site damage problem. Despite complex variable method, some other approaches have been successfully applied to solve problems of multiple cracks i.e. Zhou[12] used Fourier transform method to study two collinear symmetrical cracks, Chang[13] used numerical technique to solve two collinear cracks problem, multiple crack problem was solved by Wu[14] using weight function approach and there are many examples.

Moreover, the situation is rather more critical when stresses applied at boundary of the plate increase to an extent such that the yield zones between two neighbouring cracks get coalesced. Feng et al. [15] have examined the formation of macroscopic cracks due coalescence of two or more micro cracks. Coalescence of yield zones in collinear multiple cracks have been studied in stiffened sheet by Nishimura [16]. Furthermore, the coalescence conditions were analyzed in [17] when the adjoining plastic zones coalesced at the edge of a semi-infinite sheet. Kotousov et al. [18] studied plastic collapse between collinear cracks using distributed dislocation technique.

The prime focus of this paper is to retrace the applicability of the Dugdale model for coalescence conditions of yield zones. Also, determine load-carrying capacity of the plate containing three collinear straight cracks when existing yield zones at the internal tips of two closely situated cracks get coalesced. Parabolic stress distribution is used for different crack configuration in [19] and similar attempt has been made in [20] and [21] for linear stress distribution.

Nomenclature

$C_i (i = 0,1,2)$	constants used in the problem
E	Young’s modulus
$F(\theta, k), E(\theta, k), \Pi(\theta, \alpha^2, k)$	elliptic integral of first, second and third kind respectively
$L_i (i = 1,2,3)$	cracks
$P_n(z)$	degree n polynomial
$\pm a_1, \pm b_1, c_1, d_1$	tips of the original cracks
$\pm b, \pm a$	tips of the extended cracks
$p(t), q(t)$	stress acting on the yield zones
$z = x + iy$	complex variable

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Γ'	$-\frac{1}{2}(N_1 - N_2)e^{-2i\alpha}$, N_1 and N_2 are the principal stresses at infinity, α be the angle between N_1 and the real-axis
$p_i(i = 1,2,\ldots,5)$	denotes yield zones
$\Omega(z) = \omega'(z), \Phi(z) = \phi'(z)$	complex potential functions
γ	Poisson's ratio
μ	shear modulus
κ	$= \frac{3-\gamma}{1+\gamma}$ for the plane-stress, $= 3 - 4\gamma$ for the plane-strain
$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$	stress components
σ_∞	stresses acting at the boundary of the plate

2. Theoretical background of the problem

As per the mathematical formulation given by Muskhelishvili [22] the stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are defined in the form of two complex potential functions $\Phi(z)$ and $\Omega(z)$ as

$$\sigma_{xx} + \sigma_{yy} = 2[\Phi(z) + \overline{\Phi(z)}], \quad (1)$$

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \quad (2)$$

Equations (1) and (2) are transformed in the form of two Hilbert problems by assuming $\lim_{y\to 0} y\Phi'(t + iy) = 0$ for all t on straight cuts $L_i(i = 1,2,\ldots,n)$. These two problems are termed as

$$\Phi^+(t) + \Omega^-(t) = \sigma_{yy}^+ - i\sigma_{xy}^+, \quad (3)$$

$$\Phi^-(t) + \Omega^+(t) = \sigma_{yy}^- - i\sigma_{xy}^-, \quad (4)$$

on $\bigcup_{i=1}^{n} L_i$.

Stress distributions $\sigma_{yy}^\pm$, $\sigma_{xy}^\pm$ are prescribed on the rims of these straight cuts L_i, where superscript (-) and (+) denote the lower and upper rims of the cuts/ cracks. Hilbert problems defined in equations (3) and (4) is then solved using methodology given by Muskhelishvili [22]. The desired functions $\Phi(z)$ and $\Omega(z)$ be written as

$$\Phi(z) = \Phi_0(z) + \frac{P_n(z)}{X(z)} - \frac{1}{2}\overline{\Gamma}', \quad (5)$$

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$$\Omega(z) = \Omega_0(z) + \frac{P_n(z)}{X(z)} + \frac{1}{2}\overline{\Gamma}', \tag{6}$$

where

$$\Phi_0(z) = \frac{1}{2\pi i X(z)} \int_{\cup_{i=1}^{n} L_i} \frac{X^+(t)p(t)}{t-z} dt + \frac{1}{2\pi i} \int_{\cup_{i=1}^{n} L_i} \frac{q(t)}{t-z} dt, \tag{7}$$

$$\Omega_0(z) = \frac{1}{2\pi i X(z)} \int_{\cup_{i=1}^{n} L_i} \frac{X^+(t)p(t)}{t-z} dt - \frac{1}{2\pi i} \int_{\cup_{i=1}^{n} L_i} \frac{q(t)}{t-z} dt, \tag{8}$$

$$p(t) = \frac{1}{2}(\sigma_{yy}^+ + \sigma_{yy}^-) - \frac{i}{2}(\sigma_{xy}^+ + \sigma_{xy}^-), q(t) = \frac{1}{2}(\sigma_{yy}^- - \sigma_{yy}^-) - \frac{i}{2}(\sigma_{xy}^- - \sigma_{xy}^-), \tag{9}$$

$$X(z) = \prod_{k=1}^{n} \sqrt{z-a_k}\sqrt{z-b_k}, P_n(z) = C_0 z^n + C_1 z^{n-1} + \ldots + C_n. \tag{10}$$

Loading conditions at the boundary of the plate and condition of single-valuedness of displacement around the cracks enables to find the constants $C_0, .., C_n$.

$$2(\kappa+1)\int_{L_i} \frac{P_n(t)}{X(t)} dt + \kappa \int_{L_i} [\Phi_0^+(t) - \Phi_0^+(t)] dt + \int_{L_i} [\Omega_0^+(t) - \Omega_0^-(t)] dt = 0. \tag{11}$$

This theoretical background of the problem is totally extracted from Muskhelishvili [22] just to produce the manuscript in a self-sufficient style.

3. Statement of the main problem

Assume that three unequal collinear straight cracks L_1, L_2 and L_3 weaken an infinite isotropic elastic perfectly plastic plate, among them L_2 and L_3 are situated close to each other. These cracks occupy intervals $[-a_1, -b_1], [b_1, c_1]$ and $[d_1, a_1]$ respectively on the real axis as shown in figure-1. Infinite boundary of the plate is subjected to stress distribution, σ_∞, in the normal direction to the crack rims. This causes the opening of cracks in mode-I type deformation and formation of the yield zones at each crack tip. Yield zones developed at two adjacent tips (say c_1 and d_1) of closely situated cracks get coalesced when stresses acting at the boundary the plate increases. These yield zones are indicated by p_1, p_2, p_3, p_4 and p_5 and occupy the intervals $(-a, -a_1), (-b_1, -b), (b, b_1), (c_1, d_1)$ and (a_1, a) on the real axis. These yield zones are subjected to quadratically varying stress distribution $\frac{t^2}{a^2}\sigma_{ye}$

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(say σ), where σ_{ye} denotes yielding stress of the plate and t is any point of the rims of the yield zones.

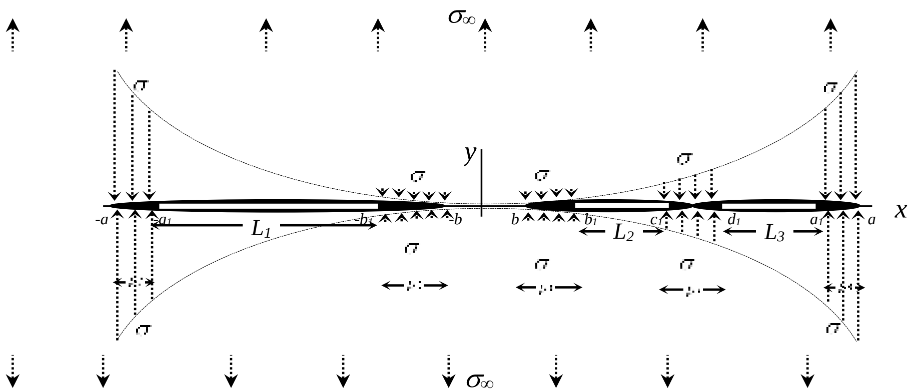

Figure-1: Pictorial representation of modified Dugdale model

4. Solution of the main problem

The above stated problem is solved by decomposing the main problem into two component problems. First component problem is an applied case which is related to applied stress, σ_∞, acting at the boundary of the plate. Second component problem is related to closing case when quadratically varying mechanical stresses is assume to be acted on the yield zones. Hence, after superposing the solutions of the component problems, the solution of the original problem will be obtained.

To illustrate the proposed method of solution, the first case related to remotely applied stresses σ_∞ of the problem as shown in figure-2. The stated problem is subjected to following boundary conditions

$$\sigma_{yy} = \sigma_\infty, \sigma_{xy} = 0, \text{when} \quad y \to \pm\infty, \tag{12}$$

$$\sigma_{yy} = 0, \sigma_{xy} = 0, \text{when} \quad y \to 0. \tag{13}$$

Using the boundary conditions given in equations (12) and (13) and the methodology shown in section-2, one can find complex potential function for two equal cracks problem or taken directly from [10],

$$\Phi_{applied}(z) = \frac{\sigma_\infty}{2\sqrt{z^2 - a^2}\sqrt{z^2 - b^2}}[z^2 - a^2\lambda^2] - \frac{\sigma_\infty}{4}. \tag{14}$$

where $\lambda^2 = \frac{E(k)}{K(k)}$, $k^2 = \frac{a^2 - b^2}{a^2}$, and $F(k), E(k)$ are the complete elliptic integral, defined by Byrd [23].

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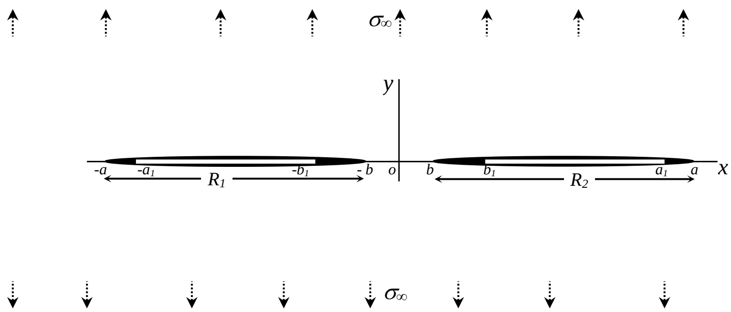

Figure-2: Opening case of equal-sized cracks

On the other hand, closing case of the problem is shown in figure-3. Consider, three unequal straight cracks with coalesced yield zones weaken an infinite isotropic elastic perfectly plastic plate. The distribution of stresses on the rims of the yield zones in the case in which yield zones between two closely placed cracks L_2 and L_3 is quadratic $\left(\frac{t^2}{a^2}\sigma_{ye}\right)$ in nature to arrest the cracks from further opening. This subproblem will be discussed under the following boundary conditions,

$$\sigma_{yy} = \frac{t^2}{a^2}\sigma_{ye}, \sigma_{xy} = 0, \quad \text{when} \quad y \to 0, x \in \bigcup_{n=1}^{4} \Gamma_n, \tag{15}$$

$$\sigma_{yy} = 0, \sigma_{xy} = 0, \qquad \text{when} \quad -\infty < x < \infty, y \to 0. \tag{16}$$

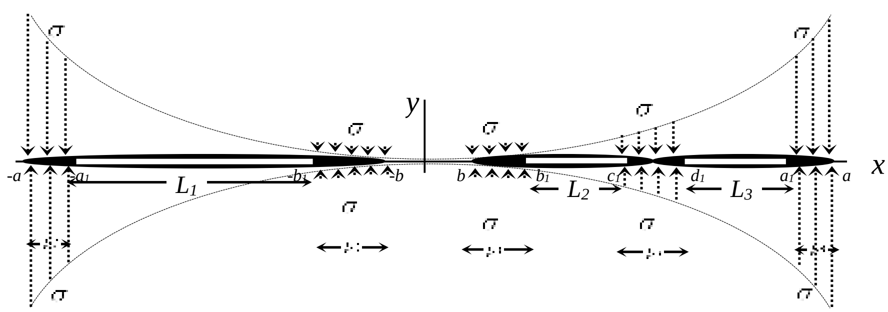

Figure-3: Closing of three unequal straight cracks

Under the boundary conditions provided in the equations (15) and (16) the solution of the problem is obtained using methodology given in Section-2. After a long mathematical calculation one can find analytical expression for function $\Phi(z)$ for the case of quadratic stress distribution,

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$$\Phi_{yield}(z) = \frac{\sigma_{ye}}{2\pi a^2 X(z)}(H_1 + 2a^4k^2G + z^2H_2 + z^2(2z^2 - a^2 - b^2)P + zH_3 + z^3H_4 + \frac{a^2 - z^2}{a}F_2(z) - 2z^2F_1(z)\sqrt{a^2 - z^2}\sqrt{z^2 - b^2}). \tag{17}$$

where

$$H_1 = \frac{(a^2 - b^2)^2}{16}S(4\theta) - \frac{(a^2 - b^2)^2}{4}P, \quad H_2 = \frac{a^2 - b^2}{2}S(2\theta),$$

$$H_3 = \frac{a^3}{3}((1 - k^2)[2F(\theta_{d_1}, k) - 2F(\theta_{c_1}, k) + E(\theta_{d_1}, k) - E(\theta_{c_1}, k)] + \frac{k^2}{2}(\sin 2\theta_{d_1}\sqrt{1 - k^2\sin^2\theta_{d_1}} - \sin 2\theta_{c_1}\sqrt{1 - k^2\sin^2\theta_{c_1}})),$$

$$H_4 = a(F(\theta_{c_1}, k) - F(\theta_{d_1}, k) + E(\theta_{c_1}, k) - E(\theta_{d_1}, k)),$$

$$S(n\theta) = \sin n\theta_{b_1} - \sin n\theta_b + 0.5\sin n\theta_{d_1} - 0.5\sin n\theta_{c_1} - \sin n\theta_{a_1},$$

$$P = \theta_{b_1} - \theta_b + 0.5\theta_{d_1} - 0.5\theta_{c_1} - \theta_{a_1},$$

$$F_2(z) = II(\theta_{c_1}, \alpha^2, k) - F(\theta_{c_1}, k) - II(\theta_{d_1}, \alpha^2, k) + F(\theta_{d_1}, k),$$

$$F_1(z) = \tanh^{-1}\left(\frac{\tan\theta_{b_1}}{\tan\theta_z}\right) - \tanh^{-1}\left(\frac{\tan\theta_b}{\tan\theta_z}\right) + 0.5\tanh^{-1}\left(\frac{\tan\theta_{d_1}}{\tan\theta_z}\right) - 0.5\tanh^{-1}\left(\frac{\tan\theta_{c_1}}{\tan\theta_z}\right) + \tanh^{-1}\left(\frac{\tan\theta_a}{\tan\theta_z}\right) - \tanh^{-1}\left(\frac{\tan\theta_{a_1}}{\tan\theta_z}\right),$$

$$G = \zeta(\theta_{b_1}, k) - \zeta(\theta_b, k) + \zeta(\theta_{d_1}, k) - \zeta(\theta_{c_1}, k) + \zeta(\theta_a, k) - \zeta(\theta_{a_1}, k) - \lambda^2(\eta(\theta_{b_1}, k) - \eta(\theta_b, k) + \eta(\theta_{d_1}, k) - \eta(\theta_{c_1}, k) + \eta(\theta_a, k) - \eta(\theta_{a_1}, k))$$

$$\zeta(\theta, k) = \left(\frac{1}{3k^2} - \frac{1}{3} + \frac{k^2}{8}\right)\theta + \frac{2 - k^2}{12}\sin 2\theta + \frac{k^2}{96}\sin 4\theta + \frac{(1 - k^2\sin^2\theta)^{\frac{3}{2}}}{3k^2}E(\theta, k)$$

$$\eta(\theta, k) = \left(\frac{1}{3k^2} - \frac{1}{6}\right)\theta + \frac{1}{12}\sin 2\theta + \frac{(1 - k^2\sin^2\theta)^{\frac{3}{2}}}{3k^2}F(\theta, k)$$

$$\sin^2\theta_t = \frac{a^2 - t^2}{a^2 - b^2}, \alpha^2 = \frac{a^2 - b^2}{a^2 - z^2}$$

5. Stress intensity factors

On superposing the solutions for two component problem given in equations (14) and (17) the mode-I type stress intensity factor may be obtained at crack tips $z = \pm a, \pm b$ using formula given by Collins [10],

$$K_I = \sqrt{8\pi}\lim_{z \to z_1}\sqrt{z - z_1}(\Phi_{applied}(z) + \Phi_{yield}(z)). \tag{18}$$

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With the help of Dugdale hypothesis, that the stress remains finite in the vicinity of the crack, four nonlinear mathematical equations are derived to calculate the yield zone length at cracks tips $t = \pm a, \pm b$ as function load ratio. These equations are as follows

$$\pi a^4(1-\lambda^2)\left(\frac{\sigma_\infty}{\sigma_{ye}}\right)_a + (H_1 + a^2H_2 + 2a^4k^2G + a^2(a^2-b^2)P + aH_3 + a^3H_4) = 0, \tag{19}$$

$$\pi a^4(1-\lambda^2)\left(\frac{\sigma_\infty}{\sigma_{ye}}\right)_{-a} + (H_1 + a^2H_2 + 2a^4k^2G + a^2(a^2-b^2)P - aH_3 - a^3H_4) = 0, \tag{20}$$

$$\pi a^2(b^2-a^2\lambda^2)\left(\frac{\sigma_\infty}{\sigma_{ye}}\right)_b + (H_1 + b^2H_2 + 2a^4k^2G + b^2(b^2-a^2)P + bH_3 + b^3H_4) = 0, \tag{21}$$

$$\pi a^2(b^2-a^2\lambda^2)\left(\frac{\sigma_\infty}{\sigma_{ye}}\right)_{-b} + (H_1 + b^2H_2 + 2a^4k^2G + b^2(b^2-a^2)P - bH_3 - b^3H_4) = 0. \tag{22}$$

These equations enable to determine yield zone length numerically.

6. Verification of analytical expressions

Analytical expressions derived for remotely applied stress ratio given in equations (19) and (20) at crack tips $t = a$ and $t = -a$ are same if $d_1 = c_1$ (because $H_3 = 0, H_4 = 0$). Furthermore, after putting $d_1 = c_1$ and $b = b_1 = 0$ in equations (19) and (20) the results transferred to the expressions given by Harrop [5] for a crack of length $2a_1$.

7. Application: yield zone length

Length of yield zones $|a - a_1|$, $|b_1 - b|$ and coalesced yield zone $|d_1 - c_1|$ are determined numerically using equations (19) to (22) under the condition that $\sigma_\infty < \sigma_{ye}$ at each crack tip. These numerical results are reported graphically in this section with respect to influencing factors like crack length, yield zone length etc.

7.1 Comparison of yield zone length at cracks tips $t = \pm a, \pm b$

Figure-4 demonstrations the change in the ratio of $\frac{b_1 - b}{a_1 - b_1}$ when applied load ratio, $\frac{\sigma_\infty}{\sigma_{ye}}$ increases to at the crack tip b and $-b$ for increasing values of $\frac{A}{D} = \frac{a_1 - b_1}{a_1 + b_1}$. Value of $\frac{A}{D}$ denotes the location of the cracks L_1 and L_2, when $\frac{A}{D} = 0.1$ the cracks are placed far away from one another and when $\frac{A}{D} = 0.9$ then they placed close to one another.

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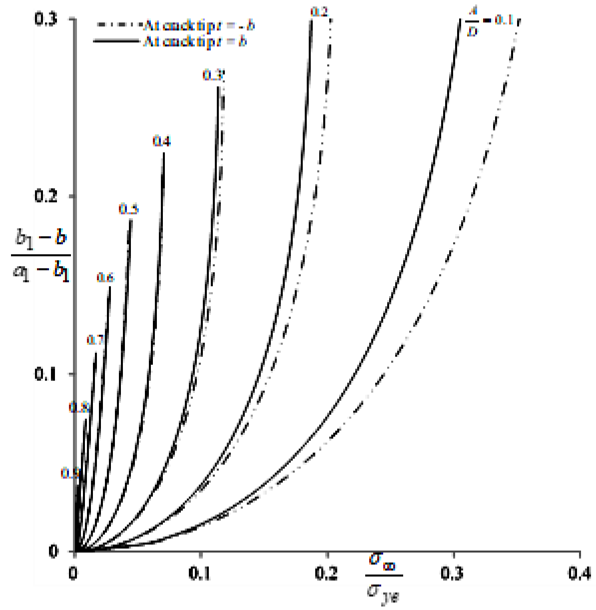

Figure-4: $\frac{\sigma_\infty}{\sigma_{ye}}$ to $\frac{b_1-b}{a_1-b_1}$

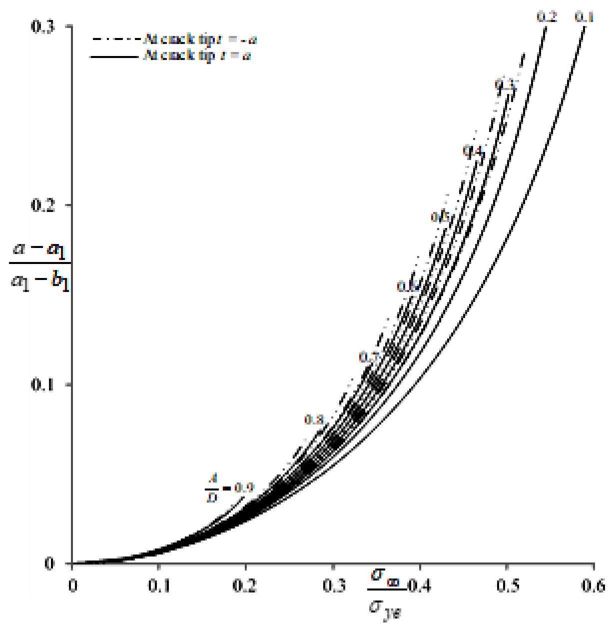

Figure-5: $\frac{\sigma_\infty}{\sigma_{ye}}$ to $\frac{a-a_1}{a_1-b_1}$

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It has been noted from the figure that the length of yield zone p_2 is smaller than the length of yield zone p_3 when $\frac{A}{D} = 0.1$. As the distance between cracks reduces the length of yield zones at both the tips, $t = b$ and $t = -b$, are same. Similar variation is plotted at the tips $t = -a$ and $t = a$ of cracks, L_1 and L_3, in figure-5. Further, the length of yield zone p_5 is smaller than the length of p_1 at $\frac{A}{D} = 0.1$ and same when $\frac{A}{D} = 0.9$.

7.2 Comparative study of yield zone length with comparable configuration of two-equal cracks

A comparative study has been carried to investigate the difference between the load-bearing capacity of an infinite plate containing three unequal cracks with coalesced yield zones (as shown in figure-1, say S_1) and when the infinite plate weakened by two-equal cracks (equivalent to configuration shown in figure-1, say S_2) under same mechanical loading conditions. These comparisons are shown in figures 6-9.

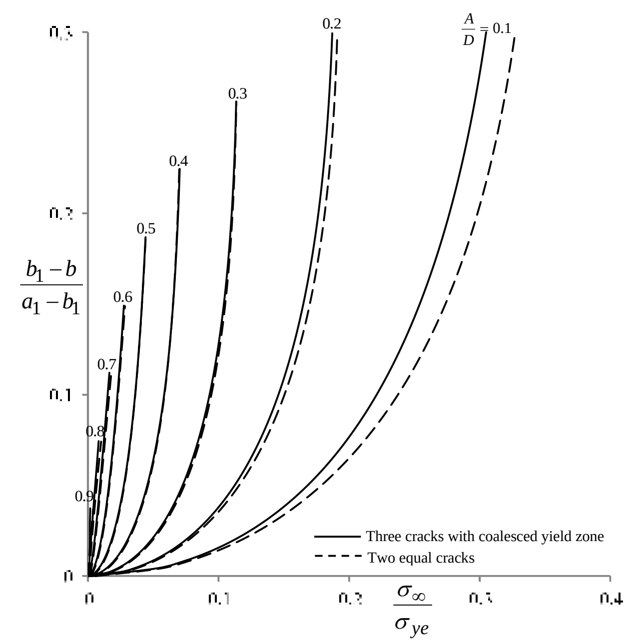

Figure-6: Variation of $\frac{\sigma_\infty}{\sigma_{ye}}$ to $\frac{b_1 - b}{a_1 - b_1}$ at crack tip b

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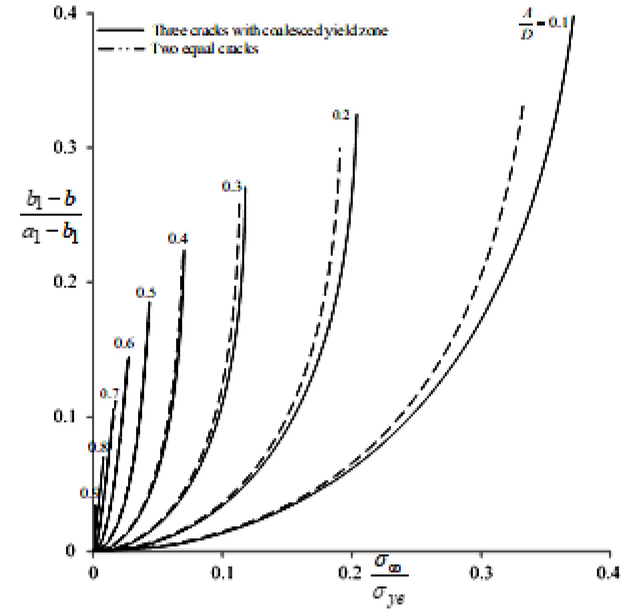

Figure-7: Variation of $\frac{\sigma_\infty}{\sigma_{ye}}$ to $\frac{b_1-b}{a_1-b_1}$ at crack tip $-b$

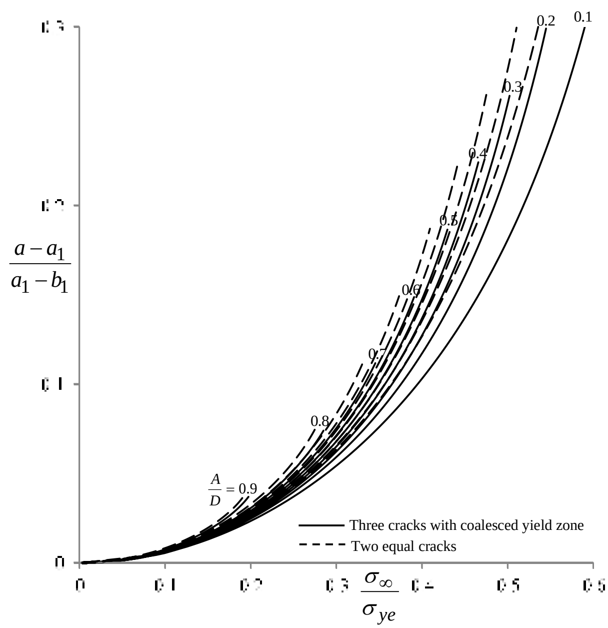

Figure-8: Variation of $\frac{\sigma_\infty}{\sigma_{ye}}$ to $\frac{a-a_1}{a_1-b_1}$ at crack tip a

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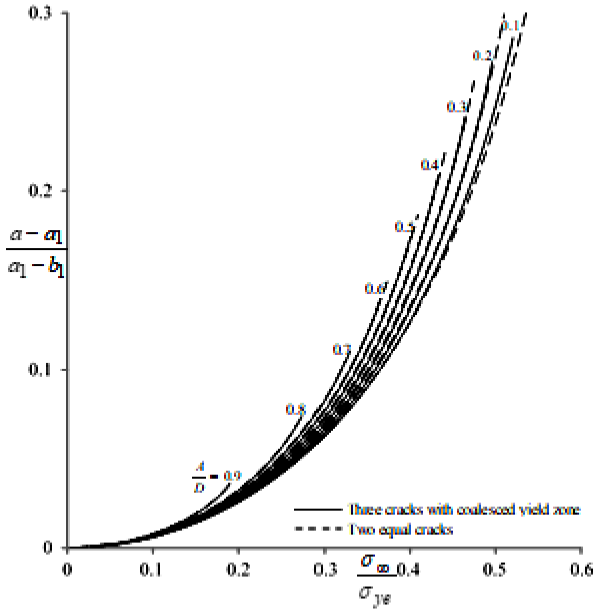

Figure-9: Variation of $\frac{\sigma_\infty}{\sigma_{ye}}$ to $\frac{a-a_1}{a_1-b_1}$ at crack tip $-a$

Variation between applied load ratio, $\frac{\sigma_\infty}{\sigma_{ye}}$ and the ratio of $\frac{b_1-b}{a_1-b_1}$ at the crack tip b is depicted in figure-6. As far as load-bearing capacity of an infinite plate is concern, the plate can tolerate more load in case of S_2 rather than S_1 at the crack tip b when $\frac{A}{D} = 0.1$. Moreover, no significant difference is seen in the said comparison when $\frac{A}{D} = 0.9$.
Figure-7 depicts the similar variation at the crack tip $t = -b$. The observations of the figure-4, the length of yield zone $|b_1 - b|$ is larger in case of S_2 as compared to S_1 when $\frac{A}{D} = 0.1$ and insignificance when $\frac{A}{D} = 0.9$. Hence, it may be concluded that coalesced yield zone is meaningful when cracks L_1 and L_2 are away from each other and insignificant when these cracks are placed close to each other. In other words, it may lose its existence.

Furthermore, analysis of the numerical results for applied load ratio $\frac{\sigma_\infty}{\sigma_{ye}}$ and the ratio of $\frac{a-a_1}{a_1-b_1}$ at the crack tip $t = a$ are shown in figure-8 for which a comparison has been made with the results of an equivalent case of two equal cracks (S_2). Under the influence of united yield zones, the bearing capacity of the plate is more with S_1 rather than S_2 for well separated cracks. In other words, the yield zone length $|p_5|$ is larger in case of S_2 in comparison to S_1 under similar mechanical loading conditions. Figure-9 shows variation

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between applied load ratio $\frac{\sigma_\infty}{\sigma_{ye}}$ and the ratio of $\frac{a-a_1}{a_1-b_1}$ at the crack tip $t=-a$. No significant difference has been observed between the said comparison at $t=-a$. Hence, less affected tip due to coalesced yield zone is the crack tip at $t=-a$.

8. Conclusion

Modification to Dugdale model has been presented for the complicated case of multiple cracks. Mathematical expressions are derived for stress intensity factors at each crack tip using complex variable method. Load-bearing capacity of the plate is evaluated numerically and the results were compared with the outcomes of two equal straight cracks. It is observed from the numerical results that coalesced yield zone is meaningful when cracks L_1 & L_2 are away from each other and meaningless when they are close to each other. Hence, it may be concluded that coalesced yield zone loses its existence. Also, the length of yield zone at the inner crack tip $t=\pm b$ is bigger than the length of yield zone at the outside crack tip $t=\pm a$.

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2. Daubechies, I., 1992, "Ten Lectures on Wavelets", *CBMS-NSF Series in Appl. Math.*, SIAM, Philadelphia, USA, pp. 23-27.

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